COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI
Vol. 33, No. 1, 1984

ed. RIKKYO UNIV/MATH
IKEBUKURO TOKYO
171 JAPAN

On the Cauchy Problem for the Wave Equation with a Singularity in the Time Variable

by

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(Received December 17, 1982)

Introduction

Let L be the first order partial differential operator defined as follows for a $\boldsymbol{C}$-valued function $u(t, x)$ of the time variable $t \in \boldsymbol{R}$ and the space variable $x = (x_1, x_2, \cdots, x_n) \in \boldsymbol{R}^n$:

$$Lu(t, x) = -\left\{\frac{a(t, x)}{t}\frac{\partial}{\partial t}u(t, x) + \boldsymbol{b}(t, x)\cdot \nabla u(t, x) + c(t, x)u(t, x)\right\},$$

where $\boldsymbol{b}(t, x) = (b_1(t, x), b_2(t, x), \cdots, b_n(t, x))$ is a $\boldsymbol{C}^n$-valued function and $\boldsymbol{b}(t, x)\cdot \nabla u(t, x) = \sum_{k=0}^{n} b_k(t, x)(\partial/\partial x_k)u(t, x)$, and $a(t, x)$ and $c(t, x)$ are $\boldsymbol{C}$-valued functions.

We are concerned with the following wave equation:

$$(\square - L)u(t, x) = f(t, x),$$

where $\square u(t, x) = (\partial^2/\partial t^2)u(t, x) - \Delta u(t, x)$. The most well-known equation of this type is the Euler-Poisson-Darboux equation (E-P-D equation) $(\square + (\lambda/t)(\partial/\partial t))u = f$, where λ is a constant. There are many works on this equation. For example, Carroll and Showalter [2], Delache and Leray [3], Diaz and Weinberger [4], and Weinstein [6]. For an extensive bibliography on the E-P-D equation we refer to [2].

We shall treat the Cauchy problem:

$$\text{(P-I)} \qquad \begin{cases} (\square - L)u(t, x) = f(t, x) \\ u(0, x) = \phi(x), \qquad \dfrac{\partial}{\partial t}u(0, x) = 0. \end{cases}$$

We have to impose on u the condition $(\partial u/\partial t)(0, x) = 0$ in order that Lu is defined in $t = 0$. Our motivation of this problem comes from a generalization of the Cauchy problem for the E-P-D equation, which was solved perfectly in [6].

It should be noted, however, that the Cauchy problem for more general partial differential operators including our L have been treated in Alinhac [1] and Tahara [5]. But we believe that our present work still has some value, because our approach is much more elementary than theirs and our results are, in fact, more detailed than theirs in some respects.

We consider (P-I) under the following assumption (A-I) on the coefficients of L throughout this article.

$$\text{(A-I)} \quad \begin{cases} (1) & a(t, x),\ b_k(t, x)\ (k=1, 2, \cdots, n) \text{ and } c(t, x) \text{ are all } \boldsymbol{C}\text{-valued} \\ & \text{functions of } (t, x) \text{ in } \boldsymbol{R}\times\boldsymbol{R}^n, \text{ and they belong to } \mathscr{B}^\infty(\boldsymbol{R}\times\boldsymbol{R}^n). \\ (2) & \text{For all positive integer } k,\ (1+a(0, x)/k)^{-1} \text{ exists and bounded.} \end{cases}$$

In order to solve (P-I) in some function spaces, we need to transform (P-I) into the auxiliary Cauchy problem (P-II):

$$\text{(P-II)} \qquad \begin{cases} (\Box - L)v_N(t, x) = F_N(t, x) \\ v_N(0, x) = \dfrac{\partial}{\partial t} v_N(0, x) = 0\,, \end{cases}$$

where F_N and v_N are derived from coefficients of L, f, ϕ and u. The natural number N is determined according to the degree of smoothness of solutions of (P-II). Roughly speaking, $F_N(t, x)$ behaves like $O(t^{N-1})$ as $t\to 0$.

Notations

$\boldsymbol{Z}_+$ denotes the set of non-negative integers, and the Greek letters $\alpha, \beta, \cdots$ denote multi-indices and are of $\boldsymbol{Z}^n_+$. $\binom{\alpha}{\gamma}$ means the multi-binomial coefficient, that is,

$$\binom{\alpha}{\gamma} = \prod_{k=1}^n \frac{\alpha_k!}{(\alpha_k-\gamma_k)!\gamma_k!}, \qquad \text{where} \quad \alpha=(\alpha_1, \alpha_2, \cdots, \alpha_n), \quad \text{and so on}.$$

$\begin{pmatrix} p\alpha\beta \\ q\gamma\delta \end{pmatrix}$ denotes $\binom{p}{q}\binom{\alpha}{\gamma}\binom{\beta}{\delta}$.

In this article we treat functions whose variables move either in $\boldsymbol{R}$ or in $\boldsymbol{R}^n$. We call an $\boldsymbol{R}$-valued variable a time variable and an $\boldsymbol{R}^n$-valued variable a space variable. A time variable will be denoted usually by t or τ. A space variable will be denoted usually by x or ξ.

D denotes the partial differentiation with respect to a time variable. ∂_k denotes the partial differentiation with respect to the k-th component of a space variable. We write $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}$.

If a function has two or more time variables $t, \tau, \cdots$, we denote the differentiation in these ones by $D_t, D_\tau, \cdots$, respectively. We use the same convention for a space variable, too.

For a function f: $\boldsymbol{R}^n\to\boldsymbol{C}$ we denote its Fourier transform by $\hat{f}$, $\mathscr{F}[f]$ or $\mathscr{F}_x[f(x)]$, and write $đx=(2\pi)^{-n/2}\,dx$. If a function f depends on a time variable as well as on a space variable, we mean by the Fourier transform of f the Fourier tansform with respect to a space variable and we write

$$\hat{f}(t, x) = \mathscr{F}[f](t, \xi) = \mathscr{F}_x[f(t, x)](\xi) = \int e^{-ix\cdot\xi} f(t, x)\,đx\,.$$

By $\|\cdot\|$ we denote the L^2-norm with respect to a space variable. But if $f\colon \boldsymbol{R}\times\boldsymbol{R}^n\to\boldsymbol{C}$ and if we want to express the dependence on t of $\|f\|$ explicitly, we write

$$\|f(t,\cdot)\| \quad\text{or}\quad \|f(t,x)\|_x$$

instead of $\|f\|$. The notation $\|f\|_x$ is also used in such cases as $\|xf(x)\|_x$. We write

$$|\nabla f|=\left(\sum_{k=1}^{n}|\partial_k f|^2\right)^{1/2}, \qquad |\nabla^2 f|=\left(\sum_{j,k=1}^{n}|\partial_j\partial_k f|^2\right)^{1/2}$$

$$\text{and}\quad \|\nabla f\|=\left(\sum_{k=1}^{n}\|\partial_k f\|^2\right)^{1/2}.$$

We denote by T an arbitrarily fixed positive number and write $I_T=(-T,T)$ and $\Omega_T=I_T\times\boldsymbol{R}^n$. If $f\colon \Omega_T\to\boldsymbol{C}$, we write $\|f\|_\infty=\sup\{|f(t,x)|;\,(t,x)\in\Omega_T\}$. If $\boldsymbol{b}=(b_1,b_2,\cdots,b_n)\colon \Omega_T\to\boldsymbol{C}^n$, we write $\|\boldsymbol{b}\|_\infty=\sum_{k=1}^n\|b_k\|_\infty$.

We shall list up the notation of function spaces used in this article. $\mathscr{S}$ denotes the set of $\boldsymbol{C}$-valued infinitely differentiable functions in $\boldsymbol{R}^n$ rapidly decreasing at infinity, and is called the Schwartz space. H^k denotes the usual Sobolev space over $\boldsymbol{R}^n$ of order k with the norm $\|f\|_k=\sum_{|\alpha|\leqq k}\|\partial^\alpha f\|$. $\mathscr{B}^k(S)$ denotes the set of all functions which have bounded continuous derivatives of order up to k in an open set $S\subset\boldsymbol{R}^n$. Let I be an interval of $\boldsymbol{R}$ and X be some function space. $C^k(I;X)$ denotes the set of all X-valued functions which have continuous derivatives of order up to k with respect to X-topology in I. If the definition domain of elements of X is S and if $f\in C^k(I;X)$, then we often write $f(t,x)$ instead of $f(t)(x)$ for $(t,x)\in I\times S$. Conversely we say that a function $f(t,x)$ of $(t,x)\in I\times S$ is in $C^k(I;X)$ if the map $I\ni t\longmapsto f(t,\cdot)$ is an element of $C^k(I;X)$. $\mathscr{B}^k(I;X)$ denotes the set of all X-valued functions which are elements of $C^k(I;X)$ and have bounded continuous derivatives of order up to k with respect to X-topology in I.

§ 1. Transform of the problem

In this section we adopt the following notation. If $w(t,x)$ is a function of $(t,x)\in\boldsymbol{R}\times\boldsymbol{R}^n$, then we write

$$w^{(k)}(x)=D^k w(0,x). \tag{1.1}$$

We suppose that f is of $C^\infty(\boldsymbol{R}\times\boldsymbol{R}^n)$ and ϕ is of $C^\infty(\boldsymbol{R}^n)$. Let a, $\boldsymbol{b}$, c are the functions of the coefficients of L.

Now we define the series of functions $\{u^{(k)}(x)\}$ successively as follows.

$$
(1.2)\begin{cases}
u^{(0)}=\phi, \qquad u^{(1)}=0, \\
u^{(2)}=(1+a^{(0)})^{-1}(f^{(0)}+\Delta\phi-b^{(0)}\cdot\nabla\phi-c^{(0)}\phi), \\
u^{(3)}=\left(1+\dfrac{a^{(0)}}{2}\right)^{-1}(f^{(1)}-a^{(1)}u^{(2)}-b^{(1)}\cdot\nabla\phi-c^{(1)}\phi), \\
u^{(4)}=\left(1+\dfrac{a^{(0)}}{3}\right)^{-1}(f^{(2)}+\Delta u^{(2)}-a^{(2)}u^{(2)}-a^{(1)}u^{(3)}-b^{(0)}\cdot\nabla u^{(2)} \\
\qquad -b^{(2)}\cdot\nabla\phi-c^{(2)}\phi-c^{(0)}u^{(2)}), \\
\text{and for } k=3,4,5,\cdots \\
u^{(k+2)}=\left(1+\dfrac{a^{(0)}}{k+1}\right)^{-1}\Big\{f^{(k)}+\Delta u^{(k)}-a^{(k)}u^{(2)}-\dfrac{k}{2}a^{(k-1)}u^{(3)}-b^{(k)}\cdot\nabla\phi-c^{(k)}\phi \\
\qquad -k!\displaystyle\sum_{m=1}^{k-2}\frac{a^{(m)}u^{(k+2-m)}}{(k+1-m)!m!}-k!\sum_{m=0}^{k-2}\frac{b^{(m)}\cdot\nabla u^{(k-m)}}{m!(k-m)!}-k!\sum_{m=0}^{k-2}\frac{c^{(m)}u^{(k-m)}}{m!(k-m)!}\Big\}.
\end{cases}
$$

Remark. If a solution u of (P-I) is sufficiently smooth, then $D^k u(0,x)$ equals $u^{(k)}(x)$. The fact is easily shown if we carry out the following calculation. Expand $u(t,x)$, $a(t,x)$, $b_k(t,x)$, $c(t,x)$ and $f(t,x)$ to the Taylor series about $t=0$ with respect to t, and insert these series into $(\square-L)u=f$, and we obtain (1.2).

LEMMA 1.1. *Each $u^{(0)}$, $u^{(1)}$, $\cdots$ is expressed as a linear form of derivatives f and ϕ whose coefficients are polynomials of $a^{(p)}$, $b_k^{(q)}$, $c^{(r)}$ and $(1+a^{(0)}/(s+1))^{-1}$ with p, q, r, and $s\in\boldsymbol{Z}_+$ as follows:*

$u^{(0)}=\phi$, $\qquad u^{(1)}=0$,

$u^{(2)}=$ *a linear form of* $\{f^{(0)}$ *and* $\partial^\alpha\phi$ *with* $|\alpha|\leqq 2\}$,

$u^{(3)}=$ *a linear form of* $\{f^{(p)}$ *with* $p=0,1$ *and* $\partial^\alpha\phi$ *with* $|\alpha|\leqq 2\}$,

$u^{(4)}=$ *a linear form of* $\{f^{(p)}$ *with* $p=0,1,2$, $\nabla f^{(0)}$, $\Delta f^{(0)}$ *and* $\partial^\alpha\phi$ *with* $|\alpha|\leqq 4\}$,

and for $k=3,4,5,\cdots$

$u^{(k+2)}=$ *a linear form of* $\{f^{(p)}$ *with* $p=0,1,\cdots,k$ *and* $\partial^\alpha f^{(q)}$ *with* $q=0,1,\cdots,k-2$ *and* $|\alpha|\leqq 2$, *and* $\partial^\beta\phi$ *with* $|\beta|\leqq k+1\}$.

Proof. The facts for $k=0,1,2,3,4$ are evident. Noting that the right-hand side of $u^{(k+2)}$ with $k\geqq 3$ contains neither $\Delta u^{(m)}$, $\nabla u^{(m)}$ with $m>k$ nor $u^{(m)}$ with $m>k+1$, we can show the facts for $k\geqq 5$ inductively.

Let N be some positive integer $\geqq 2$. We set

$$
F_N(t,x)=f(t,x)-(\square-L)\sum_{k=0}^{N}\frac{t^k}{k!}u^{(k)}(x). \tag{1.3}
$$

LEMMA 1.2. *Let* $\alpha \in \boldsymbol{Z}^n_+$, *then for* $p=0, 1, \cdots, N-2$

$$(D^p \partial^\alpha F_N)(0, x)=0\,, \quad \textit{and} \tag{1.4}$$

$$D^p \partial^\alpha F_N(t, x)=\int_0^t \frac{(t-s)^{N-2-p}}{(N-2-p)!}(D^{N-1}\partial^\alpha F_N)(s, x)ds\,. \tag{1.5}$$

Proof. $u^{(k)}/k!$ is the coefficient of t^k in the formal Taylor series about $t=0$. We remark that the forms of (1.2) are determined by the relation

$$(\square - L)\sum_{k=0}^{\infty}\frac{t^k}{k!}u^{(k)}=\sum_{k=0}^{\infty}\frac{t^k}{k}f^{(k)}\,.$$

Since

$$\left(1+\frac{a^{(0)}}{k+1}\right)u^{(k+2)}+(-\varDelta+\boldsymbol{b}^{(0)}\cdot\nabla+c^{(0)})u^{(k)}=f^{(k)}$$

holds, we have

$$\begin{aligned}F_N(t, x)=&\sum_{k=0}^{N-2}\frac{t^k}{k!}f^{(k)}+\frac{t^{N-1}}{(N-1)!}f^{(N-1)}(t', x)-\sum_{k=0}^{N-2}\left\{\left(1+\frac{a^{(0)}(x)}{k+1}\right)u^{(k+2)}(x)\right.\\&\left.+(-\varDelta+\boldsymbol{b}^{(0)}(x)\cdot\nabla+c^{(0)}(x))u^{(k)}(x)\right\}t^k/k!\\&-(-\varDelta+\boldsymbol{b}^{(0)}(x)\cdot\nabla+c^{(0)}(x))\left\{\frac{t^{N-1}}{(N-1)!}u^{(N-1)}(x)+\frac{t^N}{N!}u^{(N)}(x)\right\}\\=&\frac{t^{N-1}}{(N-1)!}f^{(N-1)}(t', x)-(-\varDelta+\boldsymbol{b}^{(0)}(x)\cdot\nabla+c^{(0)}(x))\\&\times\left\{\frac{t^{N-1}}{(N-1)!}u^{(N-1)}(x)+\frac{t^N}{N!}u^{(N)}(x)\right\},\end{aligned}$$

where t' is a number between 0 and t. Hence for any $\alpha \in \boldsymbol{Z}^n_+$, we obtain $D^p\partial^\alpha F_N(0, x)=0$ for $p=0, 1, \cdots, N-2$. Applying the Taylor formula to $D^p\partial^\alpha F_N(t, x)$, we get immediately (1.5) from (1.4), which completes the proof of the lemma.

Let $u(t, x)$ be a smooth solution of (P-I) and set

$$v_N(t, x)=u(t, x)-\sum_{k=0}^{N}\frac{t^k}{k!}u^{(k)}(x)\,. \tag{1.6}$$

We can easily show that $v_N(t, x)$ satisfies the Cauchy problem

$$\text{(P-II)}\qquad \begin{cases}(\square - L)v(t, x)=F_N(t, x)\\ v(0, x)=Dv(0, x)=0\,.\end{cases}$$

Conversely, suppose that $v(t, x)$ is a solution of (P-II). And let us set

$$u(t, x)=v(t, x)+\sum_{k=0}^{N} \frac{t^k}{k!} u^{(k)}(x).$$

Then this $u(t, x)$ satisfies the Cauchy problem (P-I). Thus, if we find a solution of the Cauchy problem (P-II), we can solve (P-I).

§ 2. Existence of a classical solution

In this section we assume that

(A-II) $$\phi \in \mathscr{S} \quad \text{and} \quad f \in C^{\infty}(\boldsymbol{R}; \mathscr{S}).$$

We omit the subscript N of F_N and v_N, if there is no confusion.

Under (A-I) (p. 20) and (A-II) we can show immediately from Lemma 1.1, (1.3) and (1.5) that F belongs to $C^{\infty}(R; \mathscr{S})$ and there exists a positive constant $\mathscr{M}_{p,\alpha,\beta}$ depending on p, α and β where $p \in \boldsymbol{Z}_+$, $p \leqq N-1$ and $\alpha, \beta \in \boldsymbol{Z}_+^n$ such that

(2.1) $$\|(ix)^{\beta} \partial^{\alpha} D^p F(t, x)\|_x \leqq \mathscr{M}_{p,\alpha,\beta} |t|^{N-1-p}.$$

Here we consider the solution of the Cauchy problem for the usual wave equation:

$$\begin{cases} \square U(t, x)=F(t, x) \\ U(0, x)=DU(0, x)=0. \end{cases}$$

It is well-known that this Cauchy problem can be solved explicitly by means of the Fourier transform and the solution $U(t, x)$ is in fact expressed by

(2.2) $$U(t, x)=\int_0^t d\tau \int e^{ix\cdot\xi} \frac{\sin(|\xi|(t-\tau))}{|\xi|} \hat{F}(\tau, \xi) d\xi.$$

This U clearly belongs to $C^{\infty}(\boldsymbol{R}; \mathscr{S})$.

We use the explicit formula (2.2) in order to construct a solution of (P-II). We adopt the successive approximation method to solve (P-II). So we define the sequence of functions $\{v^k(t, x)\}$ inductively as follows. At first we set

$$v^{-1}(t, x)=0$$

and then for $k=0, 1, 2, \cdots$ we define inductively

(2.3) $$v^k(t, x)=\int_0^t d\tau \int e^{ix\cdot\xi} \frac{\sin(|\xi|(t-\tau))}{|\xi|} (\widehat{Lv}^{k-1}+\hat{F})(\tau, \xi) d\xi.$$

Note that $v^0(t, x)=U(t, x)$.

We shall show in the following lemma that Lv^k belongs to $C^{\infty}(\boldsymbol{R}; \mathscr{S})$ in spite of the singularity at $t=0$ in L. Hence the right-hand side of (2.3) is well defined for all k and all $(t, x) \in \boldsymbol{R} \times \boldsymbol{R}^n$.

LEMMA 2.1. *Let F be the function defined by* (1.3). *Then for* $k=0, 1, 2, \cdots$

i) v^k *belongs to* $C^\infty(\boldsymbol{R}; \mathscr{S})$, *and* ii) Lv^k *belongs to* $C^\infty(\boldsymbol{R}; \mathscr{S})$ *and* iii) v^k *and* v^{k+1} *satisfy*

$$\begin{cases}\square v^{k+1}(t, x)=Lv^k(t, x)+F(t, x)\\ v^{k+1}(0, x)=Dv^{k+1}(0, x)=0\,.\end{cases}$$

Proof. If $Dv^k(t, x)/t$ is in $C^\infty(\boldsymbol{R}; \mathscr{S})$, then clearly $Lv^k(t, x)$ is also in $C^\infty(\boldsymbol{R}; \mathscr{S})$. Differentiating (2.2) in t and putting $\tau=st$, we get

$$\frac{1}{t}Dv^0(t, x)=\int_0^1 ds\int e^{ix\cdot\xi}\cos(|\xi|(1-s)t)\hat{F}(st, \xi)d\xi\,.$$

Since the map $(t, \xi)\longmapsto\int_0^1\cos(|\xi|(1-s)t)\hat{F}(st, \xi)ds$ belongs to $C^\infty(\boldsymbol{R}; \mathscr{S})$, the map $(t, x)\longmapsto Dv^0(t, x)/t=\int e^{ix\cdot\xi}d\xi\int_0^1\cos(|\xi|(1-s)t)\hat{F}(st, \xi)ds$ also belongs to $C^\infty(\boldsymbol{R}; \mathscr{S})$. Thus we have $Lv^0\in C^\infty(\boldsymbol{R}; \mathscr{S})$.

Similarly we get for general k

$$\frac{1}{t}Dv^k(t, x)=\int_0^1 ds\int e^{ix\cdot\xi}\cos(|\xi|(1-s)t)(Lv^{k-1}+\hat{F})(st, \xi)d\xi\,.$$

Therefore we set that

$$Lv^k\in C^\infty(\boldsymbol{R}; \mathscr{S})\,. \tag{2.4}$$

From (2.3) and (2.4) we get immediately $v^k\in C^\infty(\boldsymbol{R}; \mathscr{S})$. Thus we have shown i) and ii).

That $v^k(0, x)=Dv^k(0, x)=0$ is clear.

Operating D^2 and Δ to (2.3), we get easily $\square v^k=Lv^{k-1}+F$.

Thus we have completed the proof.

From (2.3) we have

$$v^0(t, x)=\int_0^t d\tau\int e^{ix\cdot\xi}\frac{\sin(|\xi|(t-\tau))}{|\xi|}\hat{F}(\tau, \xi)d\xi \tag{2.5}$$

and for $k=0, 1, 2, \cdots$

$$(v^{k+1}-v^k)(t, x)=\int_0^t d\tau\int e^{ix\cdot\xi}\frac{\sin(|\xi|(t-\tau))}{|\xi|}\mathscr{F}[L(v^k-v^{k-1})](\tau, \xi)d\xi\,. \tag{2.6}$$

Let T be an arbitrarily given positive number. We set $I_T=(-T, T)$. We shall fix a $t\in I_T$ and estimate the L^2-norm of the function $x\longmapsto(ix)^\beta\partial^\alpha D^pL(v^k-v^{k-1})(t, x)$.

LEMMA 2.2. *Suppose that* V *is a function in* $C^\infty(\boldsymbol{R}; \mathscr{S})$ *and satisfies* $D^rV(0, x)=0$ *for* $r=0, 1, 2, \cdots, k$, *where* k *is any fixed non-negative integer. Let* $\hat{V}$ *be the Fourier transform of* V *with respect to the space variable. Let*

$$W(t, x)=\int_0^t d\tau\int e^{ix\cdot\xi}\frac{\sin(|\xi|(t-\tau))}{|\xi|}\hat{V}(\tau, \xi)d\xi\,.$$

Then

$$(2.7)\qquad -D_t^k LW(t,x)=\sum_{r=0}^{k}\binom{k}{r}\int_0^t d\tau\int e^{ix\cdot\xi}\Gamma_{k-r}(t,\tau,x,\xi)D_\tau^r\hat{V}(\tau,\xi)d\xi,$$

where

$$(2.8)\qquad \Gamma_{k-r}(t,\tau,x,\xi)=\cos(|\xi|(t-\tau))D_t^{k-r}\frac{a(t,x)}{t}$$
$$+\frac{\sin(|\xi|(t-\tau))}{|\xi|}D_t^{k-r}(i\boldsymbol{b}(t,x)\cdot\xi)+\frac{\sin(|\xi|(t-\tau))}{|\xi|}D_t^{k-r}c(t,x).$$

Proof. We devide $-LW$ into three parts as follows

$$-LW(t,x)=-(L_1W(t,x)+L_2W(t,x)+L_3W(t,x)),$$

where

$$-L_1W(t,x)=\frac{a(t,x)}{t}DW(t,x),\qquad -L_2W(t,x)=\boldsymbol{b}(t,x)\cdot\nabla W(t,x)\quad\text{and}$$

$$-L_3W(t,x)=c(t,x)W(t,x).$$

We shall show

$$(2.9)\qquad -D_t^kL_1W(t,x)=\sum_{r=0}^{k}\binom{k}{r}\int_0^t d\tau\int e^{ix\cdot\xi}\cos(|\xi|(t-\tau))D_t^{k-r}\frac{a(t,x)}{t}D_\tau^r\hat{V}(\tau,\xi)d\xi.$$

In fact we have

$$-D_tL_1W(t,x)=D_t\left(\frac{a(t,x)}{t}\right)\int_0^t d\tau\int e^{ix\cdot\xi}\cos(|\xi|(t-\tau))\hat{V}(\tau,\xi)d\xi$$
$$+\frac{a(t,x)}{t}D_t\int_0^t d\tau\int e^{ix\cdot\xi}\cos(|\xi|(t-\tau))\hat{V}(\tau,\xi)d\xi$$
$$=D_t\left(\frac{a}{t}\right)\int_0^t d\tau\int e^{ix\cdot\xi}\cos(|\xi|(t-\tau))\hat{V}(\tau,\xi)d\xi$$
$$+\frac{a}{t}V(t,x)+\frac{a}{t}\int_0^t d\tau\int e^{ix\cdot\xi}D_t\cos(|\xi|(t-\tau))\hat{V}(\tau,\xi)d\xi.$$

Noting that $D_t\cos(|\xi|(t-\tau))=-D_\tau\cos(|\xi|(t-\tau))$ and that $D^rV(0,x)=0$ implies $D^r\hat{V}(0,\xi)=0$, we perform the integration by parts, and we get

$$\int_0^t d\tau\int e^{ix\cdot\xi}D_t\cos(|\xi|(t-\tau)\hat{V}(\tau,\xi)d\xi=-\int e^{ix\cdot\xi}d\xi\int_0^t(D_\tau\cos(|\xi|(t-\tau)))\hat{V}(\tau,\xi)d\tau$$
$$=-V(t,x)+\int_0^t d\tau\int e^{ix\cdot\xi}\cos(|\xi|(t-\tau))D_\tau\hat{V}(\tau,\xi)d\xi.$$

Thus we get

$$-D_tL_1W(t, x)=D_t\left(\frac{a}{t}\right)\int_0^t d\tau\int e^{ix\cdot\xi}\cos(|\xi|(t-\tau))\hat{V}(\tau, \xi)d\xi$$

$$+\frac{a}{t}\int_0^t d\tau\int e^{ix\cdot\xi}\cos(|\xi|(t-\tau))D_\tau\hat{V}(\tau, \xi)d\xi\,,$$

which proves that (2.9) is true if $k=1$. Applying the mathematical induction, we get (2.9) for $k>1$. Using

$$D_t\frac{\sin(|\xi|(t-\tau))}{|\xi|}=-D_\tau\frac{\sin(|\xi|(t-\tau))}{|\xi|}$$

and $D^r\hat{V}(0, \xi)=0$ for $r=1, 2, \cdots, k$, we get by an argument similar to that for (2.9)

$$(2.10)\qquad -D_t^kL_2W(t, x)=\sum_{r=0}^{k}\binom{k}{r}\int_0^t d\tau$$

$$\times\int e^{ix\cdot\xi}\frac{\sin(|\xi|(t-\tau))}{|\xi|}D_t^{k-r}(i\boldsymbol{b}(t, x)\cdot\xi)D_\tau^r\hat{V}(\tau, \xi)d\xi\,,$$

and

$$(2.11)\quad -D_t^kL_3W(t, x)=\sum_{r=0}^{k}\binom{k}{r}\int_0^t d\tau\int e^{ix\cdot\xi}\frac{\sin(|\xi|(t-\tau))}{|\xi|}D_t^{k-r}c(t, x)D_\tau^r\hat{V}(\tau, \xi)d\xi\,.$$

Thus combining (2.9), (2.10) and (2.11), we obtain the desired equality (2.7).

LEMMA 2.3. *Let v^k $(k=0, 1, \cdots)$ be the functions defined by* (2.3). *Then for* $p=0, 1, \cdots, N-2$, $D^pLv^k(0, x)=0$.

Proof. At first we shall show $D^pLv^0(0, x)=0$. We can put $V(t, x)=F(t, x)$ in (2.7) because of $D^pF(0, x)=0$ and $F\in C^\infty(\boldsymbol{R}; \mathscr{S})$. Then since $W(t, x)=v^0(t, x)$, we have by (2.7)

$$-D_t^pLv^0(t, x)=\sum_{r=0}^{p}\binom{p}{r}\int_0^t d\tau\int e^{ix\cdot\xi}\,\Gamma_{p-r}(t, \tau, x, \xi)D_\tau^r\hat{F}(\tau, \xi)d\xi\,.$$

Hence it is enough to show that

$$-D^pL_1v^0(0, x)=\lim_{t\to 0}\left\{\sum_{r=0}^{p}\binom{p}{r}\int_0^t d\tau\right.$$

$$\left.\times\int e^{ix\cdot\xi}\cos(|\xi|(t-\tau))D_t^{p-r}\frac{a(t, x)}{t}D_\tau^r\hat{F}(\tau, \xi)d\xi\right\}=0\,.$$

To do so it is enough to show that

$$I_0(t, x)=\int_0^t d\tau\int e^{ix\cdot\xi}\cos(|\xi|(t-\tau))\frac{1}{t^{p-r+1}}D^r\hat{F}(\tau, \xi)d\xi$$

tends to 0 as $t\to 0$. Putting $\tau = st$, we get

$$I_0(t, x) = \int_0^1 tds \int e^{ix\cdot\xi} \cos(|\xi|(1-s)t) t^{-p+r-1} D^r \hat{F}(st, \xi) d\xi$$

$$= t^{-p+r} \int_0^1 ds \int e^{ix\cdot\xi} \cos(|\xi|(1-s)t) D^r \hat{F}(st, \xi) d\xi .$$

But we see from (1.5) and (A-II) that $D^r F(st, \xi) = O((st)^{N-1-r})$ withe respect to the topology of $\mathscr{S}$ in a neighbourhood of $st=0$. Hence $\lim_{t\to 0} I_0(t, x) = 0$. Thus we have shown for $p=0, 1, \cdots, N-2$ that $D^p L v^0(0, x) = 0$.

Next we shall show $D^p L(v^1 - v^0)(0, x) = 0$. By an argument similar to the case of $D^p L v^0(0, x)$, it is enough to show that

$$I_1 = t^{-p+r} \int_0^1 ds \int e^{ix\cdot\xi} \cos(|\xi|(1-s)t) D^r \widehat{Lv}^0(st, \xi) d\xi$$

tends to 0 as $t\to 0$. But, looking at the proof of $D^p L v^0(0, x) = 0$, we know further that $D^p L v^0(t, x) = O(t^{N-1-p})$ and $D^p \widehat{Lv}^0(t, \xi) = O(t^{N-1-p})$ as $t\to 0$ with respect to the topology of $\mathscr{S}$. Thus we have $I_1 \to 0$ as $t\to 0$. Consequently we have $D^p L(v^1 - v^0)(0, x) = 0$.

Repeating the same argument, we obtain $D^p L(v^k - v^{k-1})(0, x) = 0$ for $k=1, 2, \cdots$, which completes the proof.

COROLLARY 2.4. *Let $\{v^k\}$ be the sequence of the functions defined by* (2.3). *Then for $p=0, 1, \cdots, N-2$ and $k=1, 2, 3, \cdots$*

$$(2.12) \quad -D_t^p L(v^k - v^{k-1})(t, x) = \sum_{r=0}^{p} \binom{p}{r} \int_0^t d\tau \times \int e^{ix\cdot\xi} \Gamma_{p-r}(t, \tau, x, \xi) D^r \mathscr{F}[L(v^{k-1} - v^{k-2})](\tau, \xi) d\xi .$$

Proof. We have already shown in Lemma 2.1 and Lemma 2.3 that $L(v^k - v^{k-1}) \in C^\infty(\boldsymbol{R}; \mathscr{S})$ and for $p=0, 1, \cdots, N-2$ $D^p L(v^k - v^{k-1})(0, x) = 0$. Hence, noting (2.6), we get (2.12) immediately from Lemma 2.2.

LEMMA 2.5. *Let α and $\beta \in \boldsymbol{Z}_+^n$ and let $p=0, 1, \cdots, N-2$. Then for $k=0, 1, 2, \cdots$*

$$-(ix)^\beta \partial_x^\alpha D_t^p L(v^k - v^{k-1})(t, x) = (-1)^{|\beta|} \sum_{r,\gamma,\delta}^{p,\alpha,\beta} \binom{p\alpha\beta}{r\gamma\delta} \int_0^t d\tau \times \int e^{ix\cdot\xi} \{\partial_\xi^{\beta-\delta} \partial_x^\gamma \Gamma_{p-r}(t, \tau, x, \xi)\} \times \{\partial_\xi^\delta (i\xi)^{\alpha-\gamma} D_\tau^r \mathscr{F}[L(v^{k-1} - v^{k-2})](\tau, \xi)\} d\xi ,$$

where $L(v^{-1} - v^{-2})$ denotes F and $v^{-1} = 0$.

Proof. We shall show the proof for $k \geqq 1$. The proof for $k=0$ is performed similarly.

By the Leibniz formula we get

$$\partial_x^\alpha[e^{ix\cdot\xi}\Gamma_{p-r}(t,\tau,x,\xi)] = \sum_{\gamma\leqq\alpha}\binom{\alpha}{\gamma}(\partial_x^{\alpha-\gamma}e^{ix\cdot\xi})(\partial_x^\gamma\Gamma_{p-r}(t,\tau,x,\xi))$$

$$=\left\{\sum_{\gamma\leqq\alpha}\binom{\alpha}{\gamma}(i\xi)^{\alpha-\gamma}\partial_x^\gamma\Gamma_{p-r}\right\}e^{ix\cdot\xi}.$$

Hence we get from Corollary 2.4

$$\begin{aligned}(2.13)\qquad & -(ix)^\beta\partial_x^\alpha D_t^p L(v^k-v^{k-1})(t,x) \\ &= \sum_{r=0}^{p}\sum_{\gamma\leqq\alpha}\binom{p}{r}\binom{\alpha}{\gamma}(ix)^\beta\int_0^t d\tau \\ &\quad\times\int e^{ix\cdot\xi}(\partial_x^\gamma\Gamma_{p-r})(i\xi)^{\alpha-\gamma}D_\tau^r\mathscr{F}[L(v^{k-1}-v^{k-2})](\tau,\xi)d\!\!{}^-\xi.\end{aligned}$$

Noting that the map $\xi \longmapsto D_\tau^r\mathscr{F}[L(v^{k-1}-v^{k-2})](\tau,\xi)$ belongs to $\mathscr{S}$ for any fixed τ, we can integrate (2.13) by parts in ξ. And we get

$$\begin{aligned}-(ix)^\beta\partial_x^\alpha D_t^p L(v^k-v^{k-1})(t,x) &= \sum_{r=0}^{p}\sum_{\gamma\leqq\alpha}\binom{p}{r}\binom{\alpha}{\gamma}\int_0^t d\tau \\ &\quad\times\int(\partial_\xi^\beta e^{ix\cdot\xi})(\partial_x^\gamma\Gamma_{p-r}(t,\tau,x,\xi))(i\xi)^{\alpha-\gamma} \\ &\quad\times D_\tau^r\mathscr{F}[L(v^{k-1}-v^{k-2})](\tau,\xi)d\!\!{}^-\xi \\ &=(-1)^{|\beta|}\sum_{r=0}^{p}\sum_{\gamma\leqq\alpha}\binom{p}{r}\binom{\alpha}{\gamma}\int_0^t d\tau \\ &\quad\times\int e^{ix\cdot\xi}\partial_\xi^\beta\{\partial_x^\gamma\Gamma_{p-r}(t,\tau,x,\xi)(i\xi)^{\alpha-\gamma} \\ &\quad\times D_\tau^r\mathscr{F}[L(v^{k-1}-v^{k-2})](\tau,\xi)\}d\!\!{}^-\xi.\end{aligned}$$

Applying the Leibniz formula again, we get

$$\begin{aligned}-(ix)^\beta\partial_x^\alpha D_t^p L(v^k-v^{k-1})(t,x) &=(-1)^{|\beta|}\sum_{r=0}^{p}\sum_{\gamma\leqq\alpha}\binom{p}{r}\binom{\alpha}{\gamma}\sum_{\delta\leqq\beta}\binom{\beta}{\delta}\int_0^t d\tau \\ &\quad\times\int e^{ix\cdot\xi}(\partial_\xi^{\beta-\delta}\partial_x^\gamma\Gamma_{p-r})\partial_\xi^\delta\{(i\xi)^{\alpha-\gamma} \\ &\quad\times D_\tau^r\mathscr{F}[L(v^{k-1}-v^{k-2})](\tau,\xi)\}d\!\!{}^-\xi \\ &=(-1)^{|\beta|}\sum_{r,\gamma,\delta}^{p,\alpha,\beta}\binom{p\alpha\beta}{r\gamma\delta}\int_0^t d\tau\int e^{ix\cdot\xi}(\partial_\xi^{\beta-\delta}\partial_x^\gamma\Gamma_{p-r}) \\ &\quad\times\partial_\xi^\delta\{(i\xi)^{\alpha-\gamma}D_\tau^r\mathscr{F}[L(v^{k-1}-v^{k-2})](\tau,\xi)\}d\!\!{}^-\xi,\end{aligned}$$

which completes the proof.

LEMMA 2.6. *Let T be an arbitrarily fixed positive constant and $\alpha \in \boldsymbol{Z}^n_+$. Then there exists a positive constant $C_{\alpha,T}$ depending only on α and T such that, if $t\tau \geqq 0$ and $|\tau| \leqq |t| \leqq T$,*

$$\text{i)}\quad \left|\partial^\alpha \frac{\sin(|\xi|(t-\tau))}{|\xi|}\right| \leqq C_{\alpha,T}, \qquad \text{ii)}\quad |\partial^\alpha \cos(|\xi|(t-\tau))| \leqq C_{\alpha,T} \quad \textit{and}$$

$$\text{iii)}\quad \sum_{k=1}^n \left|\partial^\alpha \frac{\xi_k \sin(|\xi|(t-\tau))}{|\xi|}\right| \leqq C_{\alpha,T}.$$

Proof. We shall write a proof of i) only. Other cases are proved similarly. $\sin(|\xi|t)/|\xi|$ is infinitely differentiable in $(t, \xi) \in \boldsymbol{R} \times \boldsymbol{R}^n$. Since $\partial^\alpha\{\sin(|\xi|t)/|\xi|\}$ is continuous in the compact domain $\{(t, \xi); |t| \leqq T \text{ and } |\xi| \leqq 1\}$, there exists a positive constant $C'_{\alpha,T}$ such that, if $|t| \leqq T$ and $|\xi| \leqq 1$, $|\partial^\alpha \sin(|\xi|t)/|\xi|| \leqq C'_{\alpha,T}$.

Next we consider the case of $|t| \leqq T$ and $|\xi| \geqq 1$. By the Leibniz formula we get

$$\partial^\alpha \frac{\sin(|\xi|t)}{|\xi|} = \sum_{\beta \leqq \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} \sin(|\xi|t) \partial^\beta (|\xi|)^{-1}.$$

Hence there exists a positive constant $C''_{\alpha,T}$ such that, if $|t| \leqq T$ and $|\xi| \geqq 1$, $|\partial^\alpha \sin(|\xi|t)/|\xi|| \leqq C''_{\alpha,T}$.

Thus setting $C_{\alpha,T} = \max(C'_{\alpha,T}, C''_{\alpha,T})$, we have for any $t, \tau \in \boldsymbol{R}$ with $|\tau| \leqq |t| \leqq T$ and for any $\xi \in \boldsymbol{R}^n$

$$\left|\partial^\alpha \frac{\sin(|\xi|(t-\tau))}{|\xi|}\right| \leqq C_{\alpha,T}.$$

Thus we have got the desired estimate i).

Here we define several constants to be used in the following lemma.

Let α be any fixed element of $\boldsymbol{Z}^n_+$. We set

$$K_{\alpha,T} = \max C_{\alpha',T} \qquad \text{for all} \quad \alpha' \leqq \alpha. \tag{2.14}$$

For $p \in \boldsymbol{Z}_+$, $\alpha \in \boldsymbol{Z}^n_+$ and $T > 0$ we let

$$C(p, \alpha, T) = \max\{1, \|t^{r+1}\partial^{\alpha'} D^r(at^{-1})\|_\infty, \|t^{r+1}\partial^{\alpha'} D^r(ib_k)\|_\infty, \|t^{t+1}\partial^{\alpha'} D^r c\|_\infty\}, \tag{2.15}$$

where max. is taken for all $\alpha' \leqq \alpha$, $r = 0, 1, \cdots, p$ and $k = 1, 2, \cdots, n$.

Let α and $\beta \in \boldsymbol{Z}^n_+$, and $p \in \boldsymbol{Z}_+$. We set

$$\begin{aligned} a_{p,\alpha,\beta,T} = {} & 3(n+2)(p+1)C(p, \alpha, T)^2 K^2_{\beta,T} \\ & \times \left(\prod_{k=1}^n (\alpha_k+1)(\beta_k+1)\right) \frac{(2\alpha)!}{(\alpha!)^2} \frac{(2\beta)!}{(\beta!)^2} \frac{(2p)!}{(p!)^2}. \end{aligned} \tag{2.16}$$

We set

$$M_{p,\alpha,\beta}=\max(m_{p',\alpha',\beta'}), \tag{2.17}$$

where max. is taken for all $p'\leqq p$, $\alpha'\leqq\alpha$ and $\beta'\leqq\beta$, and $m_{p,\alpha,\beta}$ is the constant appearing in (2.1).

We remark that the following elementary inequality holds:

$$\left(\sum_{k=1}^{m} a_k\right)^2 \leqq m\sum_{k=1}^{m} a_k^2 . \tag{2.18}$$

LEMMA 2.7. *Let $p, p'\in \boldsymbol{Z}_+$ and $\alpha, \alpha', \beta, \beta'\in \boldsymbol{Z}^n_+$. Suppose that $p'\leqq p\leqq N-2$, $\alpha'\leqq\alpha$ and $\beta'\leqq\beta$. If $(t, x)\in\Omega_T$, then for $k=0, 1, 2, \cdots$*

$$\|(ix)^{\beta'}\partial^{\alpha'}D^{p'}L(v^k-v^{k-1})(t, x)\|_x^2\leqq\begin{cases}\left(\dfrac{a_{p,\alpha,\beta,T}}{2N-1-2p}\right)^{k+1}M^2_{p,\alpha,\beta}t^{2N-2-2p} & \text{if } |t|\leqq 1,\\[2ex] \left(\dfrac{a_{p,\alpha,\beta,T}}{2N-1-2p}\right)^{k+1}M^2_{p,\alpha,\beta}t^{2N-2} & \text{if } |t|\geqq 1.\end{cases}$$

Proof. We shall prove the case of $|t|\leqq 1$ only, since the proof for the case of $|t|\geqq 1$ is similar and rather easier.

At first consider the case of $k=0$. From Lemma 2.5 we have

$$-(ix)^{\beta'}\partial_x^{\alpha'}D_t^{p'}Lv^0(t, x)=(-1)^{|\beta'|}\sum_{r,\gamma,\delta}^{p',\alpha',\beta'}\begin{pmatrix}p' & \alpha' & \beta'\\ r & \gamma & \delta\end{pmatrix}\int_0^t d\tau$$

$$\times\int e^{ix\cdot\xi}(\partial_\xi^{\beta'-\delta}\partial_x^{\gamma}\Gamma_{p'-r})\partial_\xi^{\delta}((i\xi)^{\alpha'-\gamma}D_\tau^r\hat{F}(\tau, \xi))d\xi .$$

Using the Schwarz inequality and (2.18), we get

$$\|(ix)^{\beta'}\partial^{\alpha'}D^{p'}Lv^0(t, x)\|_x^2$$

$$\leqq|t|\int dx\left|\int_0^t d\tau\left|\int e^{ix\cdot\xi}\sum_{r,\gamma,\delta}^{p',\alpha',\beta'}\begin{pmatrix}p' & \alpha' & \beta'\\ r & \gamma & \delta\end{pmatrix}(\partial_\xi^{\beta'-\delta}\partial_x^{\gamma}\Gamma_{p-r})\right.\right.$$

$$\left.\left.\times(\partial_\xi^{\delta}((i\xi)^{\alpha'-\gamma}D_\tau^r\hat{F}(\tau, \xi))d\xi\right|^2\right|$$

$$\leqq(p'+1)\prod_{k=1}^{n}(\alpha'_k+1)(\beta'_k+1)\sum_{r,\gamma,\delta}^{p',\alpha',\beta'}\begin{pmatrix}p' & \alpha' & \beta'\\ r & \gamma & \delta\end{pmatrix}^2$$

$$\times|t|\left|\int_0^t d\tau\int dx\left|\int e^{ix\cdot\xi}(\partial_\xi^{\beta'-\delta}\partial_x^{\gamma}\Gamma_{p'-r})(\partial_\xi^{\delta}((i\xi)^{\alpha'-\gamma}D_\tau^r\hat{F}(\tau, \xi))d\xi\right|^2\right|$$

$$=(p'+1)\prod_{k=1}^{n}(\alpha'_k+1)(\beta'_k+1)\sum_{r,\gamma,\delta}^{p',\alpha',\beta'}\begin{pmatrix}p' & \alpha' & \beta'\\ r & \gamma & \delta\end{pmatrix}^2|t|\left|\int_0^t d\tau\int dx\right|\left(\partial_x^{\gamma}D_t^{p'-r}\frac{a}{t}\right)$$

$$\times\int e^{ix\cdot\xi}\{\partial_\xi^{\beta'-\delta}\cos(|\xi|(t-\tau))\}\{\partial_\xi^{\delta}((i\xi)^{\alpha'-\gamma}D_\tau^r\hat{F}(\tau, \xi))\}d\xi$$

$$+\sum_{k=1}^{n}(\partial_x^{\gamma}D_t^{p'-r}ib_k(t,x))\int e^{ix\cdot\xi}\left\{\partial_\xi^{\beta'-\delta}\frac{\xi_k\sin(|\xi|(t-\tau))}{|\xi|}\right\}\{\partial_\xi^{\delta}((i\xi)^{\alpha'-\gamma}D_\tau^{r}\hat{F}(\tau,\xi)\}d\xi$$

$$+(\partial_x^{\gamma}D_t^{p'-r}c(t,x))\int e^{ix\cdot\xi}\left\{\partial_\xi^{\beta'-\delta}\frac{\sin(|\xi|(t-\tau))}{|\xi|}\right\}\{\partial_\xi^{\delta}((i\xi)^{\alpha'-\gamma}D_\tau^{r}\hat{F}(\tau,\xi))\}d\xi\Big|^2\Big|.$$

Using (2.18) again for the inside of the above $|\cdots|^2$ which contains $n+2$ terms, and referring to (2.15), we have

$$\|(ix)^{\beta'}\partial^{\alpha'}D^{p'}Lv^0(t,x)\|_x^2$$

$$\leqq(p'+1)\prod_{k=1}^{n}(\alpha'_k+1)(\beta'_k+1)\sum_{r,\gamma,\delta}^{p',\alpha',\beta'}\begin{pmatrix}p' & \alpha' & \beta'\\ r & \gamma & \delta\end{pmatrix}^2|t|\left|\int_0^t d\tau\int dx\,(n+2)\right.$$

$$\times\left[\left|\partial^{\gamma}D_t^{p'-r}\frac{a}{t}\right|^2\left|\int e^{ix\cdot\xi}\{\partial_\xi^{\beta'-\delta}\cos(|\xi|(t-\tau))\}\{\partial_\xi^{\delta}((i\xi)^{\alpha'-\gamma}D_\tau^{r}\hat{F}(\tau,\xi))\}d\xi\right|^2\right.$$

$$+\sum_{k=1}^{n}|\partial^{\gamma}D^{p'-r}ib_k|^2\left|\int e^{ix\cdot\xi}\left\{\partial_\xi^{\beta'-\delta}\frac{\xi_k\sin(|\xi|(t-\tau))}{|\xi|}\right\}(\partial_\xi^{\delta}((i\xi)^{\alpha'-\gamma}D_\tau^{r}\hat{F}(\tau,\xi))\}d\xi\right|^2$$

$$\left.\left.+|\partial^{\gamma}D^{p'-r}c|^2\left|\int e^{ix\cdot\xi}\left\{\partial_\xi^{\beta'-\delta}\frac{\sin(|\xi|(t-\tau))}{|\xi|}\right\}\{\partial_\xi^{\delta}((i\xi)^{\alpha'-\gamma}D_\tau^{r}\hat{F}(\tau,\xi))\}d\xi\right|^2\right]\right|$$

$$\leqq(n+2)(p'+1)\prod_{k=1}^{n}(\alpha'_k+1)(\beta'_k+1)$$

$$\times\sum_{r,\gamma,\delta}^{p',\alpha',\beta'}\begin{pmatrix}p' & \alpha' & \beta'\\ r & \gamma & \delta\end{pmatrix}^2|t|\frac{C(p,\alpha,T)^2}{t^{2p'-2r+2}}\left|\int_0^t[I_1+I_2+I_3]\,d\tau\right|,$$

where

$$I_1=\int dx\left|\int e^{ix\cdot\xi}\{\partial_\xi^{\beta'-\delta}\cos(|\xi|(t-\tau))\}\{\partial_\xi^{\delta}((i\xi)^{\alpha'-\gamma}D_\tau^{r}\hat{F}(\tau,\xi))\}d\xi\right|^2,$$

$$I_2=\sum_{k=1}^{n}\int dx\left|\int e^{ix\cdot\xi}\left\{\partial_\xi^{\beta'-\delta}\frac{\xi_k\sin(|\xi|(t-\tau))}{|\xi|}\right\}\{\partial_\xi^{\delta}((i\xi)^{\alpha'-\gamma}D_\tau^{r}\hat{F}(\tau,\xi))\}d\xi\right|^2$$

and

$$I_3=\int dx\left|\int e^{ix\cdot\xi}\left\{\partial_\xi^{\beta'-\delta}\frac{\sin(|\xi|(t-\tau))}{|\xi|}\right\}\{\partial_\xi^{\delta}((i\xi)^{\alpha'-\gamma}D_\tau^{r}\hat{F}(\tau,\xi))\}d\xi\right|^2.$$

Using Lemma 2.6 and applying the Plancherel theorem again, we get for $j=1, 2, 3$

$$I_j\leqq C_{\beta'-\delta,T}^2\int|\partial_\xi^{\delta}(i\xi)^{\alpha'-\gamma}D^r\hat{F}(\tau,\xi))|^2d\xi=C_{\beta'-\delta,T}^2\int|(ix)^{\delta}\partial_x^{\alpha'-\gamma}D^rF(\tau,x)|^2dx$$

Thus we get

$$\|(ix)^{\beta'}\partial^{\alpha'}D^{p'}Lv^0(t,x)\|_x^2$$
$$\leqq (n+2)(p'+1)\prod_{k=1}^{n}(\alpha'_k+1)(\beta'_k+1)\sum_{r,\gamma,\delta}^{p,\alpha,\beta}\begin{pmatrix}p' & \alpha' & \beta'\\ r & \gamma & \delta\end{pmatrix}^2\frac{C(p,\alpha,T)^2}{|t|^{2p'-2r+1}}$$
$$\times\left|\int_0^t[3C^2_{\beta'-\delta,T}\int|(ix)^{\delta}\partial^{\alpha'-\gamma}D^rF(\tau,x)|^2dx]d\tau\right|.$$

From (2.14) we have $K_{\beta,T}\geqq C_{\beta'-\delta,T}$. Thus we get

$$\|(ix)^{\beta'}\partial^{\alpha'}D^{p'}Lv^0(t,x)\|_x^2$$
$$\leqq 3(n+2)(p+1)\prod_{k=1}^{n}(\alpha_k+1)(\beta_k+1)\sum_{r,\gamma,\delta}^{p',\alpha',\beta'}\begin{pmatrix}p' & \alpha' & \beta'\\ r & \gamma & \delta\end{pmatrix}^2C(p,\alpha,T)^2K^2_{\beta,T}$$
$$\times|t|^{-2p'+2r-1}\left|\int_0^t d\tau\|(ix)^{\delta}\partial^{\alpha'-\gamma}D^rF(\tau,x)\|_x^2\right|.$$

Now from (2.1) we get

$$\|(ix)^{\beta'}\partial^{\alpha'}D^{p'}Lv^0(t,x)\|_x^2$$
$$\leqq 3(n+2)(p+1)\prod_{k=1}^{n}(\alpha_k+1)(\beta_k+1)\sum_{r,\gamma,\delta}^{p',\alpha',\beta'}\begin{pmatrix}p' & \alpha' & \beta'\\ r & \gamma & \delta\end{pmatrix}^2C(p,\alpha,T)^2K_{\beta,T}$$
$$\times\frac{m^2_{\alpha'-\gamma,\delta,p'}}{2N-1-2r}\cdot\frac{|t|^{2N-1-2r}}{|t|^{2p'-2r+1}}.$$

From (2.17) we have $M_{\alpha,\beta,p}\geqq m_{\alpha'-\gamma,\delta,p'}$. And the following fact holds:

$$\sum_{r,\gamma,\delta}^{p',\alpha',\beta'}\begin{pmatrix}p' & \alpha' & \beta'\\ r & \gamma & \delta\end{pmatrix}^2\leqq\sum_{r,\gamma,\delta}^{p,\alpha,\beta}\begin{pmatrix}p & \alpha & \beta\\ r & \gamma & \delta\end{pmatrix}^2=\frac{(2p)!(2\alpha)!(2\beta)!}{(p!)^2(\alpha!)^2(\beta!)^2}$$

for $p'\leqq p$, $\alpha'\leqq\alpha$ and $\beta'\leqq\beta$. Thus we have

$$(2.19)\qquad \|(ix)^{\beta'}\partial^{\alpha'}D^{p'}Lv^0(t,x)\|_x^2$$
$$\leqq 3(n+2)(p+1)\prod_{k=1}^{n}(\alpha_k+1)(\beta_k+1)\cdot\frac{(2p)!(2\alpha)!(2\beta)!}{(p!)^2(\alpha!)^2(\beta!)^2}$$
$$\times C(p,\alpha,T)^2K^2_{\beta,T}\frac{M^2_{p,\alpha,\beta}}{2N-1-2p}t^{2N-2-2p'}$$
$$=\frac{a_{p,\alpha,\beta,T}}{2N-1-2p}M^2_{p,\alpha,\beta}t^{2N-2-2p'}.$$

Here, if $|t|\leqq 1$, we have $t^{-2p'}\leqq t^{-2p}$ for $p'\leqq p$. Thus we have

$$\|(ix)^{\beta'}\partial^{\alpha'}D^{p'}Lv^0(t,x)\|_x^2\leqq\frac{a_{p,\alpha,\beta,T}}{2N-1-2p}M^2_{p,\alpha,\beta}t^{2N-2-2p},$$

which complete the proof for $k=0$.

Next consider the case of $k=1$. By the same calculation as the preceding one we get

$$\|(ix)^{\beta'}\partial^{\alpha'}D^{p'}L(v^1-v^0)(t, x)\|_x^2$$
$$\leqq 3(n+2)(p+1)\prod_{k=1}^{n}(\alpha_k+1)(\beta_k+1)\sum_{r,\gamma,\delta}^{p,\alpha,\beta}\begin{pmatrix}p & \alpha & \beta\\ r & \gamma & \delta\end{pmatrix}^2 C(p, \alpha, T)^2 K_{\beta,T}^2$$
$$\times |t|^{-2p'+2r-1}\left|\int_0^t \|(ix)^{\delta}\partial^{\alpha'-\gamma}D^r Lv^0(\tau, x)\|_x^2\, d\tau\right|.$$

Noting that $r\leqq p'$, $\alpha'-\gamma\leqq\alpha'\leqq\alpha$ and $\delta\leqq\beta'\leqq\beta$, we get from (2.19)

$$\|(ix)^{\beta'}\partial^{\alpha'}D^{p'}L(v^1-v^0)(t, x)\|_x^2\leqq\frac{(a_{p,\alpha,\beta,T})^2}{2N-1-2p}M_{p,\alpha,\beta}^2\cdot(2N-1-2r)^{-1}\frac{|t|^{2N-1-2r}}{|t|^{2p'-2r+1}}$$
$$\leqq\left(\frac{a_{p,\alpha,\beta,T}}{2N-1-2p}\right)^2 M_{p,\alpha,\beta}^2 t^{2N-2-2p}.$$

Repeating the same argument successively, we obtain for $k=0, 1, 2, \cdots$

$$\|(ix)^{\beta'}\partial^{\alpha'}D^{p'}L(v^k-v^{k-1})(t, x)\|_x^2\leqq\left(\frac{a_{p,\alpha,\beta,T}}{2N-1-2p}\right)^{k+1}\times M_{p,\alpha,\beta}^2 t^{2N-2-2p}.$$

Thus we have completed the proof.

LEMMA 2.8. *Let α and β be any elements of $\boldsymbol{Z}_+^n$ and p be any non-negative integer $\leqq N-2$. If $(t, x)\in\Omega_T$, then the following estimate holds:*

$$\|(ix)^{\beta}\partial^{\alpha}D^p(v^k-v^{k-1})(t, x)\|_x^2\leqq\begin{cases} M_{p,\alpha,\beta}^2\left(\dfrac{a_{p,\alpha,\beta,T}}{2N-1-2p}\right)^{k+1} t^{2N-2p} & \text{if } |t|\leqq 1,\\[2ex] M_{p,\alpha,\beta}^2\left(\dfrac{a_{p,\alpha,\beta,T}}{2N-1-2p}\right)^{k+1} t^{2N} & \text{if } |t|>1,\end{cases}$$

for $k=0, 1, 2, \cdots$.

Proof. We showed in Lemma 2.1 that $Lv^k\in C^\infty(\boldsymbol{R}; \mathscr{S})$ and in Lemma 2.3 that $D^pLv^k(0, x)=0$ for $p=0, 1, \cdots, N-2$. Hence by the same calculation as in Lemma 2.2 we get

$$\int_0^t\left(D_t\frac{\sin(|\xi|(t-\tau))}{|\xi|}\right)D_\tau^{p-1}\mathscr{F}[L(v^{k-1}-v^{k-2})](\tau, \xi)d\tau$$
$$=\int_0^t\frac{\sin(|\xi|(t-\tau))}{|\xi|}D_\tau^p\mathscr{F}[L(v^{k-1}-v^{k-2})](\tau, \xi)d\tau.$$

Therefore we get

$$(2.20)\quad (ix)^{\beta}\partial^{\alpha}D^{p}(v^{k}-v^{k-1})(t,x)=(ix)^{\beta}\partial^{\alpha}D^{p}\int_{0}^{t}d\tau$$
$$\times\int e^{ix\cdot\xi}\frac{\sin(|\xi|(t-\tau))}{|\xi|}\mathscr{F}[L(v^{k-1}-v^{k-2})](\tau,\xi)d\xi$$
$$=(-1)^{|\beta|}\sum_{\delta\leqq\beta}\binom{\beta}{\delta}\int_{0}^{t}d\tau\int e^{ix\cdot\xi}\partial_{\xi}^{\beta-\delta}\left(\frac{\sin(|\xi|(t-\tau))}{|\xi|}\right)$$
$$\times\partial_{\xi}^{\alpha}\{(i\xi)^{\alpha}D_{\tau}^{p}\mathscr{F}[L(v^{k-1}-v^{k-2})](\tau,\xi)\}d\xi,$$

where we have used the fact that $D^{p}\mathscr{F}[L(v^{k-1}-v^{k-2})](\tau,\xi)$ is in $C^{\infty}(\boldsymbol{R};\mathscr{S})$, and the integration by parts and the Leibniz formula.

To prove the lemma we apply the Schwarz inequality and the Plancherel theorem to (2.20), and we use (2.18). Then we have

$$(2.21)\quad \|(ix)^{\beta}\partial^{\alpha}D^{p}(v^{k}-v^{k-1})(t,x)\|_{x}^{2}$$
$$\leqq|t|\int dx\left|\int_{0}^{t}d\tau\left|\int e^{ix\cdot\xi}\sum_{\delta\leqq\beta}\binom{\beta}{\delta}\left\{\partial_{\xi}^{\beta-\delta}\frac{\sin(|\xi|(t-\tau))}{|\xi|}\right\}\right.\right.$$
$$\left.\left.\times\{\partial_{\xi}^{\delta}(i\xi)^{\alpha}D_{\tau}^{p}\mathscr{F}[L(v^{k-1}-v^{k-2})](\tau,\xi)\}d\xi\right|^{2}\right|$$
$$\leqq\prod_{k=1}^{n}(\beta_{k}+1)\sum_{\delta\leqq\beta}\binom{\beta}{\delta}^{2}|t|\left|\int_{0}^{t}d\tau\int dx\right.$$
$$\left.\times\left|\int e^{ix\cdot\xi}\partial_{\xi}^{\beta-\delta}\frac{\sin(|\xi|(t-\tau))}{|\xi|}\partial_{\xi}^{\delta}\{(i\xi)^{\alpha}D_{\tau}^{p}\mathscr{F}[L(v^{k-1}-v^{k-2})](\tau,\xi)d\xi\right|^{2}\right.$$
$$=\prod_{k=1}^{n}(\beta_{k}+1)\sum_{\delta\leqq\beta}\binom{\beta}{\delta}^{2}|t|\left|\int_{0}^{t}d\tau\right.$$
$$\left.\times\left\|\partial_{\delta}^{\beta-\delta}\frac{\sin(|\xi|(t-\tau))}{|\xi|}\partial_{\xi}^{\delta}\{(i\xi)^{\alpha}D_{\tau}^{p}\mathscr{F}[L(v^{k-1}-v^{k-2})](\tau,\xi)\}\right\|^{2}\right|.$$

Using the Plancherel theorem again and Lemma 2.6 and (2.14), we have

$$(2.22)\quad \left\|\partial_{\xi}^{\beta-\delta}\left\{\frac{\sin(|\xi|(t-\tau))}{|\xi|}\right\}\partial_{\xi}^{\delta}\{(i\xi)^{\alpha}D^{p}\mathscr{F}[L(v^{k-1}-v^{k-2})](\tau,\xi)\}\right\|_{\xi}^{2}$$
$$\leqq C_{\beta-\delta,T}^{2}\|\partial^{\delta}\{(i\xi)^{\alpha}D^{p}\mathscr{F}[L(v^{k-1}-v^{k-2})](\tau,\xi)\|_{\xi}^{2}$$
$$\leqq K_{\beta,T}^{2}\|(ix)^{\delta}\partial^{\alpha}D^{p}L(v^{k-1}-v^{k-2})(\tau,x)\|_{x}^{2}.$$

Consider only the case of $|t|\leqq 1$ as in the previous lemma. From Lemma 2.7 we have

(2.23) $$\|(ix)^{\delta}\partial^{\alpha}D^{p}L(v^{k-1}-v^{k-2})(\tau, x)\|_x^2$$
$$\leqq \left(\frac{a_{p,\alpha,\beta,T}}{2N-1-2p}\right)^k M_{p,\alpha,\beta}^2 \tau^{2N-2-2p} \qquad \text{for } |\tau|\leqq|t|\leqq 1 .$$

Thus combining (2.21), (2.22) and (2.23), we obtain

$$\|(ix)^{\beta}\partial^{\alpha}D^{p}(v^{k}-v^{k-1})(t, x)\|_x^2$$
$$\leqq \prod_{k=1}^{n}(\beta_k+1)\sum_{\delta\leqq\beta}\binom{\beta}{\delta}^2 K_{\beta,T}^2\left(\frac{a_{p,\alpha,\beta,T}}{2N-1-2p}\right)^k M_{p,\alpha,\beta}^2|t|\left|\int_0^t \tau^{2N-2-2p}d\tau\right|$$
$$=M_{p,\alpha,\beta}^2\left(\frac{a_{p,\alpha,\beta,T}}{2N-1-2p}\right)^k K_{\beta,T}^2\prod_{k=1}^{n}(\beta_k+1)\cdot\frac{(2\beta)!}{(\beta!)^2}\cdot\frac{t^{2N-2p}}{2N-1-2p}.$$

In view of the definition (2.16) of $a_{p,\alpha,\beta,T}$ we have

$$a_{p,\alpha,\beta,T}\geqq K_{\beta,T}^2\prod_{k=1}^{n}(\beta_k+1)\cdot\frac{(2\beta)!}{(\beta!)^2}.$$

Thus we have shown

$$\|(ix)^{\beta}\partial^{\alpha}D^{p}(v^{k}-v^{k-1})(t, x)\|_x^2\leqq M_{p,\alpha,\beta}^2\left(\frac{a_{p,\alpha,\beta,T}}{2N-1-2p}\right)^{k+1} t^{2N-2p}.$$

Until now we have made the estimates of $v^k(t, x)$ by the L^2-norm in x for a fixed t. But in order to obtain solutions of (P-II) in $C^\infty(\boldsymbol{R}; \mathscr{S})$, we need to estimate $v^k(t, x)$ by the maximum norm in x. For that purpose, using the same process as in the Sovolev imbedding theorem, we turn the L^2-estimate into the maximum estimates by means of the Fourier transform.

Remark. All constants except $C_{\gamma,\delta}$ in the following lemma and its proof depend on T and the dimensional number n of $\boldsymbol{R}^n$. But for the sake of simplicity we omit T and n from the subscripts of those constants except for c_n in the proof. Therefore, for example, $a_{p,\alpha,\beta}$ will stand for $a_{p,\alpha,\beta,T}$ which is used in Lemma 2.7 and Lemma 2.8.

In the following e denotes $(1, 1, \cdots, 1)\in \boldsymbol{Z}_+^n$.

LEMMA 2.9. *Let $p\in Z_+$ and $\alpha, \beta\in \boldsymbol{Z}_+^n$. Let $m=[n/4]+1$. Suppose that N is a natural number $\geqq p+2$. Then there exist some positive constants $A_{p,\alpha,\beta}$ and $B_{p,\alpha,\beta}$ such that for $k=0, 1, 2, \cdots$*

(i) $|(ix)^{\beta}\partial^{\alpha}D^{p}(v^{k}-v^{k-1})(t, x)|$

$$\leqq\begin{cases} A_{p,\alpha,\beta}\left(\dfrac{a_{p,\alpha+2me,\beta}}{2N-1-2p}\right)^{(k+1)/2}|t|^{N-p} & \text{if } |t|\leqq 1\\[2ex] A_{p,\alpha,\beta}\left(\dfrac{a_{p,\alpha+2me,\beta}}{2N-1-2p}\right)^{(k+1)/2}|t|^{N} & \text{if } |t|>1,\end{cases}$$

(ii) $|\partial^{\beta}(i\xi)^{\alpha}D^{p}\mathscr{F}[L(v^{k}-v^{k-1})](t, \xi)|$

$$\leqq\begin{cases} B_{p,\alpha,\beta}\left(\dfrac{a_{p,\alpha,\beta+2me}}{2N-1-2p}\right)^{(k+1)/2}|t|^{N-1-p} & \text{if } |t|\leqq 1 \\ B_{p,\alpha,\beta}\left(\dfrac{a_{p,\alpha,\beta+2me}}{2N-1-2p}\right)^{(k+1)/2}|t|^{N-1} & \text{if } |t|>1 . \end{cases}$$

Proof. As was shown in Lemma 2.1, the function $x \longmapsto (ix)^{\beta}\partial^{\alpha}D^{p}(v^{k}-v^{k-1})(t, x)$ belongs to $\mathscr{S}$ for any fixed t. Applying the Fourier transform and the Schwarz inequality, we get

$$\begin{aligned} |(ix)^{\beta}\partial^{\alpha}D^{p}(v^{k}-v^{k-1})(t, x)| &= \left|\int e^{ix\cdot\xi}\mathscr{F}_{x}[(ix)^{\beta}\partial^{\alpha}D^{p}(v^{k}-v^{k-1})(t, x)](\xi)d\xi\right| \\ &\leqq c_{n}\|(1+|\xi|^{2})^{m}\mathscr{F}_{x}[(ix)^{\beta}\partial^{\alpha}D^{p}(v^{k}-v^{k-1})(t, x)](\xi)\|_{\xi} \\ &= c_{n}\|(1-\Delta)^{m}(ix)^{\beta}\partial^{\alpha}D^{p}(v^{k}-v^{k-1})(t, x)\|_{x} \\ &\leqq c_{n}\sum_{r=0}^{m}\binom{m}{r}\|\Delta^{r}(ix)^{\beta}\partial^{\alpha}D^{p}(v^{k}-v^{k-1})(t, x)\|_{x}, \end{aligned}$$

where

$$c_{n}=(1/2\pi)^{n/2}\left\{\int(1+|\xi|^{2})^{-2m}d\xi\right\}^{1/2} .$$

Further we can find positive constants $C_{\gamma,\delta}$ depending only on γ and $\delta\in \boldsymbol{Z}_{+}^{n}$ such that

$$\|\Delta^{r}(ix)^{\beta}\partial^{\alpha}D^{p}(v^{k}-v^{k-1})(t, x)\|_{x}\leqq \sum_{\gamma,\delta}^{\alpha+2re,\beta} C_{\gamma,\delta}\|(ix)^{\delta}\partial^{\gamma}D^{p}(v^{k}-v^{k-1})(t, x)\|_{x}. \tag{2.24}$$

But, if $|t|\leqq 1$, we have from Lemma 2.8

$$\|(ix)^{\delta}\partial^{\gamma}D^{p}(v^{k}-v^{k-1})(t, x)\|_{x}\leqq M_{p,\alpha+2me,\beta}\left(\frac{a_{p,\alpha+2me,\beta}}{2N-1-2p}\right)^{(k+1)/2}|t|^{N-p}$$

for $\gamma\leqq\alpha+2me$ and $\delta\leqq\beta$. Hence we have

$$\begin{aligned} &|(ix)^{\beta}\partial^{\alpha}D^{p}(v^{k}-v^{k-1})(t, x)| \\ &\leqq c_{n}\sum_{r=0}^{m}\binom{m}{r}\sum_{\gamma,\delta}^{\alpha+2re,\beta} C_{\gamma,\delta}\|(ix)^{\delta}\partial^{\gamma}D^{p}(v^{k}-v^{k-1})(t, x)\|_{x} \\ &\leqq c_{n}M_{p,\alpha+2me,\beta}\sum_{r=0}^{m}\binom{m}{r}\sum_{\gamma,\delta}^{\alpha+2re,\beta} C_{\gamma,\delta}\left(\frac{a_{p,\alpha+2me,\beta}}{2N-1-2p}\right)^{(k+1)/2}|t|^{N-p}. \end{aligned}$$

Setting

$$A_{p,\alpha,\beta}=c_{n}M_{p,\alpha+2me,\beta}\sum_{r=0}^{m}\binom{m}{r}\sum_{\gamma,\delta}^{\alpha+2re,\beta} C_{\gamma,\delta},$$

we obtain the estimate (i) for $|t|\leqq 1$.

In order to obtain (ii) let

$$B_{p,\alpha,\beta}=c_n M_{p,\alpha,\beta+2re}\sum_{r=0}^{m}\binom{m}{r}\sum_{\gamma,\delta}^{\alpha,\beta+2re}C_{\gamma,\delta},$$

where $C_{\gamma,\delta}$ is the same constant as in (2.24).

Let us perform the following calculation similar to the above ones. Then we get

$$\begin{aligned}(2.25)\quad &|\partial_\xi^\beta(i\xi)^\alpha D^p\mathscr{F}[L(v^k-v^{k-1})](t,\xi)|\\ &=\left|\int e^{-ix\cdot\xi}(ix)^\beta\partial^\alpha D^pL(v^k-v^{k-1})(t,x)dx\right|\\ &\leqq c_n\|(1+|x|^2)^m(ix)^\beta\partial_x^\alpha D^pL(v^k-v^{k-1})(t,x)\|_x\\ &\leqq c_n\sum_{r=0}^{m}\binom{m}{r}\sum_{\gamma,\delta}^{\alpha,\beta+2re}C_{\gamma,\delta}\|\partial_\xi^\delta(i\xi)^\gamma D^p\mathscr{F}[L(v^k-v^{k-1})](t,\xi)\|_\xi\\ &=c_n\sum_{r=0}^{m}\binom{m}{r}\sum_{\gamma,\delta}^{\alpha,\beta+2re}C_{\gamma,\delta}\|(ix)^\delta\partial_x^\gamma D^pL(v^k-v^{k-1})(t,x)\|_x,\end{aligned}$$

from which the estimate (ii) for $|t|\leqq 1$ follows immediately.

We can obtain the estimates for the case of $|t|>1$ by calculations which are similar to and rather easier than those for the case of $|t|\leqq 1$.

Thus all the proof has been completed.

Let p, q and r be non-negative integers. We denote by $a(p,q,r)$

$$(2.26)\qquad \text{Max}\,\{a_{p,\alpha,\beta,T};\ |\alpha|\leqq(q+2m)n,\ |\beta|\leqq(r+2m)n\}.$$

Until now we have not discussed whether the sequence of functions $\{v^k(t,x)\}$ converges or not as $k\to\infty$. We have left a natural number N indefinite. Looking at the definitions of $a_{p,\alpha,\beta,T}$ by (2.16) and $a(p,q,r)$ by the above (2.26), we see that these numbers are independent of N, whence we can determine N for given p, q, $r\in\boldsymbol{Z}_+$ and T so that

$$(2.27)\qquad p+2\leqq N \quad\text{and}\quad 0<\frac{a(p,q,r)}{2N-1-2p}<1.$$

Let N satisfy (2.27). If $p'\leqq p$, $q'\leqq q$ and $r'\leqq r$, then

$$a(p',q',r')\leqq a(p,q,r),$$

in view of (2.16), and hence

$$(2.28)\qquad 0<\frac{a(p',q',r')}{2N-1-2p'}<1.$$

We define $F_N(t,x)$ for this N by (1.3), and then define $v_N^k(t,x)$ for this $F_N(t,x)$ by (2.3). We consider the sequence of functions $\{x^\beta\partial^\alpha D^p v_N^k(t,x)\}$.

LEMMA 2.10. *Let p, q and r be arbitrarily given natural numbers $\geqq 2$ and N be a natural number satisfying* (2.27).

(i) *If $p' \in \boldsymbol{Z}_+$ satisfies $p' \leqq p$ and $\alpha, \beta \in \boldsymbol{Z}_+^n$ satisfy $|\alpha| \leqq q$ and $|\beta| \leqq r$, respectively, then the sequence $v_N^k(t, x)$ $(k=0, 1, 2, \cdots)$ converges as $k \to \infty$ uniformly in Ω_T. Set*

$$v_N(t, x) = \lim_{k \to \infty} v_N^k(t, x) . \tag{2.29}$$

Then the sequence $x^\beta \partial^\alpha D^{p'} v_N^k(t, x)$ $(k=0, 1, 2, \cdots)$ also converges as $k \to \infty$ uniformly in Ω_T, and

$$\lim_{k \to \infty} x^\beta \partial^\alpha D^{p'} v_N^k(t, x) = x^\beta \partial^\alpha D^{p'} v_N(t, x) ,$$

and $x^\beta \partial^\alpha D^{p'} v_N(t, x)$ is defined as a bounded continuous function of $(t, x) \in \Omega_T$ and satisfies

$$|x^\beta \partial^\alpha D^{p'} v_N(t, x)| \leqq \begin{cases} C|t|^{N-p} & \text{if } |t| \leqq 1 \\ C|t|^N & \text{if } |t| > 1 , \end{cases} \tag{2.30}$$

where C is a positive constant depending only on p, q, r, T and n.

(ii) *$v_N(t, x)$ is a solution of the Cauchy problem*

$$\begin{cases} (\Box - L) v_N(t, x) = F_N(t, x) , \\ v_N(0, x) = D v_N(0, x) = 0 \end{cases} \tag{2.31}$$

(iii) *If $q, r \geqq n+1$, then $v_N(t, x)$ satisfies*

$$v_N(t, x) = \int_0^t d\tau \int e^{ix \cdot \xi} \frac{\sin (|\xi|(t-\tau))}{|\xi|} (\widehat{Lv_N} + \widehat{F_N})(\tau, \xi) d\xi .$$

Proof. Let $p' \leqq p$, $|\alpha| \leqq q$ and $|\beta| \leqq r$. Then it follows from (i) of Lemma 2.9 and (2.27) that the sequence $x^\beta \partial^\alpha D^{p'} v_N^k(t, x)$ $(k=0, 1, 2, \cdots)$ converges as $k \to \infty$ to some function $W_{p', \alpha, \beta}(t, x)$ uniformly with respect to $(t, x) \in \Omega_T$. Therefore, noting the definition (2.29), $x^\beta \partial^\alpha D^{p'} v_N(t, x)$ is well defined and equals $W_{p', \alpha, \beta}(t, x)$ for $(t, x) \in \Omega_T$. This implies further that $x^\beta \partial^\alpha D^{p'} v_N(t, x)$ is continuous in (t, x) and satisfies the estimate (2.30), where C is defined as

$$C = \max \{A_{p, \alpha, \beta}; |\alpha| \leqq q, |\beta| \leqq r\} \times \left(\frac{\dfrac{a(p, q, r)}{2N-1-2p}}{1 - \dfrac{a(p, q, r)}{2N-1-2p}} \right)^{1/2} .$$

In particular it follows from (2.30) that $x^\beta \partial^\alpha D^{p'} v_N(t, x)$ is a bounded function of $(t, x) \in \Omega_T$. This completes the proof of the assertion (i) of the lemma.

In order to prove our assertion (ii) we show next that $Lv_N(t, x)$ makes sense and the equality

$$Lv_N(t, x) = \lim_{k \to \infty} Lv_N^k(t, x) \tag{2.32}$$

holds for $(t, x) \in \Omega_T$. In fact this is trivial for $t \neq 0$ by virtue of the proof of the assertion (i). On the other hand, noting that $N>3$, it follows from (2.30) that

$$\lim_{t\to 0} \frac{a(t, x)}{t} Dv_N(t, x) = a(0, x) \lim_{t\to 0} \frac{Dv_N(t, x)}{t} = a(0, x) D^2 v_N(0, x) = 0 .$$

Hence $Lv_N(0, x)$ make sense and equals 0. It follows from this and Lemma 2.3 that (2.32) holds for all $(t, x) \in \Omega_T$. $v_N^k(t, x)$ and $v_N^{k+1}(t, x)$ satisfy the equality $\Box v_N^{k+1}(t, x) - Lv_N^k(t, x) = F_N(t, x)$, which was shown in Lemma 2.1. Now we can let $k \to \infty$ in the above equality. That $v_N(0, x) = Dv_N(0, x) = 0$ is already shown in (2.30). Thus we see that $v_N(t, x)$ satisfies (2.31).

In order to prove our assertion (iii), assume first the inequality $r \geqq n+1$ only. Then Lemma 2.9, (i) and (2.27) and (2.30) imply that $(1+|x|)^{n+1} v_N(t, x)$, $(1+|x|)^{n+1} \partial_j v_N^k(t, x)$ $(j=1, 2, \cdots, n)$ and $(1+|x|)^{n+1} Dv_N^k(t, x)/t$ converge as $k \to \infty$ uniformly in Ω_T to $(1+|x|)^{n+1} v_N(t, x)$, $(1+|x|)^{n+1} \partial_j v_N(t, x)$ and $(1+|x|)^{n+1} Dv_N(t, x)/t$, respectively. This means that $(1+|x|)^{n+1} Lv_N^k(t, x)$ converges uniformly in Ω_T to $(1+|x|)^{n+1} Lv_N(t, x)$. Therefore $Lv_N(t, x)$ is summable over $\boldsymbol{R}^n$ for any fixed t and the Fourier transform $\widehat{Lv_N}(t, \xi)$ exists, and so the equality

$$\widehat{Lv_N}(t, \xi) = \lim_{k\to\infty} \widehat{Lv_N^k}(t, \xi) \tag{2.33}$$

holds for $(t, \xi) \in \Omega_T$.

Finally assume further the inequality $q \geqq n+1$. Then Lemma 2.9, (ii) and (2.27) and (2.33) imply that $(1+|\xi|)^{n+1} \widehat{Lv_N^k}(t, \xi)$ converges as $k \to \infty$ uniformly in Ω_T to $(1+|\xi|)^{n+1} \widehat{Lv_N}(t, \xi)$. Therefore we have

$$v_N(t, x) = \lim_{k\to\infty} v_N^k(t, x) = \int_0^t d\tau \int e^{ix\cdot\xi} \frac{\sin(|\xi|(t-\tau))}{|\xi|} (\widehat{Lv_N} + \widehat{F_N})(\tau, \xi) d\xi .$$

This completes the proof of the lemma.

At length we can show now that the problem (P-I) has a solution in $C^\infty(\boldsymbol{R}; \mathscr{S})$.

THEOREM 2.11. *Suppose that* (A-I) *and* (A-II). *Then the Cauchy problem* (P-I) *has a unique solution in* $C^\infty(\boldsymbol{R}; \mathscr{S})$.

Proof. For any sufficiently large positive number T it is enough to show that (P-I) has a unique solution in $\mathscr{B}^\infty(I_T; \mathscr{S})$.

By Lemma 2.10 we know that, given three natural numbers $p \geqq 2$, $q \geqq 2$ and $r \geqq 2$, there are an N and $v_N(t, x)$ corresponding to N such that $(ix)^\beta \partial^\alpha D^{p'} v_N(t, x)$ is a bounded continuous function of $(t, x) \in \Omega_T$, if $p' \leqq p$, $|\alpha| \leqq q$ and $|\beta| \leqq r$.

On the other hand $u^{(1)}(x), u^{(2)}(x), \cdots, u^{(N)}(x)$ are constructed in terms of a, $\boldsymbol{b}$, c, ϕ and f in accordance with (1.2), and it is clear that the function

$$\sum_{k=0}^N \frac{t^k}{k!} u^{(k)}(x)$$

of $(t, x)\in\Omega_T$ belongs to the space $C^\infty(\boldsymbol{R}; \mathscr{S})$.

From our discussion at the end of the preceding section we know that, if $v_N(t, x)$ is a solution of the problem (P-II), then

$$u(t, x)=v_N(t, x)+\sum_{k=0}^{N}\frac{t^k}{k!}u^{(k)}(x) \tag{2.34}$$

gives a solution of the Cauchy problem (P-I).

On the other hand it will be shown by Theorem 3.4 in the next section that (P-I) can have at most one solution under more general conditions on ϕ and f than those of this section. So the solution $u(t, x)$ of (P-I) constructed above dose not actually depend on N. Therefore, taking N as large as one likes, we see that $(ix)^\beta\partial^\alpha D^p u(t, x)$ is a bounded continuous function of $(t, x)\in\Omega_T$ for any α, $\beta\in\boldsymbol{Z}^n_+$ and $p\in\boldsymbol{Z}_+$, which means that the solution u of (P-I) constructed by (2.34) belongs to $\mathscr{B}^\infty(I_T; \mathscr{S})$. We have completed the proof of the theorem under the assumption that Theorem 3.4 is true.

COROLLARY 2.12. *Let $v_N(t, x)$ be the solution of* (P-II) *constructed in Lemma* 2.10. *Then v_N belongs to $C^\infty(\boldsymbol{R}; \mathscr{S})$.*

Proof. By Theorem 2.11 we already know that there exists a solution u of (P-I) which belongs to $C^\infty(\boldsymbol{R}; \mathscr{S})$. In terms of this solution u of (P-I) we can write

$$v_N(t, x)=u(t, x)-\sum_{k=0}^{N}\frac{t^k}{k!}u^{(k)}(x).$$

But the function

$$\sum_{k=0}^{N}\frac{t^k}{k!}u^{(k)}(x)$$

belongs to $C^\infty(\boldsymbol{R}; \mathscr{S})$. Therefore v_N also belongs to $C^\infty(\boldsymbol{R}; \mathscr{S})$.

§ 3. Energy estimates and uniqueness

In this section we shall prove three kinds of energy inequalities for (P-II). Using these energy inequalities, we shall prove the uniqueness theorem for (P-I) which was needed in the proof of Theorem 2.11 and will be used in that of the existence theorem of strong solutions of (P-I) in the next section.

Let M be a natural number $\geqq 2$. Let smooth functions f: $\boldsymbol{R}\times\boldsymbol{R}^n\to\boldsymbol{C}$ and ϕ: $\boldsymbol{R}^n\to\boldsymbol{C}$ be given. Let T be an arbitrarily given positive number. We suppose that a solution $u(t, x)$ of (P-I) exists and is in $C^{M+1}(\Omega_T)\cap\mathscr{B}^{M+1}(I_T; H^k)$, where $k=1$ or 2. We set

$$F_M(t, x)=f(t, x)-(\Box-L)\sum_{k=0}^{M}\frac{t^k}{k!}u^{(k)}(x), \tag{3.1}$$

where $u^{(k)}$'s are the functions given in Lemma 1.1 in terms of f, ϕ and their derivatives.

LEMMA 3.1. *Define the functions* $(t, x) \longmapsto G_M(f, \phi)(t, x)$ *and* $(t, x) \longmapsto H_M(f, \phi)(t, x)$ *as follows.*

$$G_2(f, \phi)(t, x)=|Df(t, x)|^2+\sum_{|\alpha|\leqq 2}|\partial^\alpha f(0, x)|^2+\sum_{|\alpha|\leqq 4}|\partial^\alpha\phi(x)|^2,$$

$$G_3(f, \phi)(t, x)=|D^2f(t, x)|^2+\sum_{|\alpha|\leqq 2}\sum_{p=0}^{1}|\partial^\alpha D^p f(0, x)|^2+\sum_{|\alpha|\leqq 4}|\partial^\alpha\phi(x)|^2,$$

$$G_4(f, \phi)(t, x)=|D^3f(t, x)|^2+\sum_{|\alpha|\leqq 4}\sum_{p=0}^{2}|\partial^\alpha D^p f(0, x)|^2+\sum_{|\alpha|\leqq 6}|\partial^\alpha\phi(x)|^2,$$

$$H_2(f, \phi)(t, x)=\sum_{k=1}^{n}|\partial_k Df(t, x)|^2+\sum_{|\alpha|\leqq 3}|\partial^\alpha f(0, x)|^2+\sum_{|\alpha|\leqq 5}|\partial^\alpha\phi(x)|^2,$$

$$H_3(f, \phi)(t, x)=\sum_{k=1}^{n}|\partial_k D^2f(t, x)|^2+\sum_{|\alpha|\leqq 3}\sum_{p=0}^{1}|\partial^\alpha D^p f(0, x)|^2+\sum_{|\alpha|\leqq 5}|\partial^\alpha\phi(x)|^2,$$

$$H_4(f, \phi)(t, x)=\sum_{k=1}^{n}|\partial_k D^3f(t, x)|^2+\sum_{|\alpha|\leqq 5}\sum_{p=0}^{2}|\partial^\alpha D^p f(0, x)|^2+\sum_{|\alpha|\leqq 7}|\partial^\alpha\phi(x)|^2$$

and for $M\geqq 5$

$$G_M(f, \phi)(t, x)=|D^{M-1}f(t, x)|^2+\sum_{|\alpha|\leqq 4}\sum_{p=0}^{M-2}|\partial^\alpha D^p f(0, x)|^2+\sum_{|\alpha|\leqq M+1}|\partial^\alpha\phi(x)|^2,$$

$$H_M(f, \phi)(t, x)=\sum_{k=1}^{n}|\partial_k D^{M-1}f(t, x)|^2+\sum_{|\alpha|\leqq 5}\sum_{p=0}^{M-2}|\partial^\alpha D^p f(0, x)|^2+\sum_{|\alpha|\leqq M+2}|\partial^\alpha\phi(x)|^2.$$

Then there exists a positive constant C_M *depending only on* M *and* T *such that*

$$\|D^jF_M(t, \cdot)\|^2\leqq C_M|t|^{2M-3-2j}\left|\int_0^t\int G_M(f, \phi)(s, x)dx\,ds\right|$$

for $j=0, 1$, *and*

$$\|\nabla F_M(t, \cdot)\|^2\leqq C_M|t|^{2M-3}\left|\int_0^t\int H_M(f, \phi)(s, x)dx\,ds\right|.$$

Proof. Operating D^{M-1} to F_M of (3.1) and then ∇ to $D^{M-1}F_M$, we get

(3.2) $$D^{M-1}F_M(t, x)=D^{M-1}f(t, x)+\Delta u^{(M-1)}(x)+t\Delta u^{(M)}(x)$$
$$-\left\{\sum_{k=2}^{M}\frac{1}{(k-1)!}D^{M-1}(at^{k-2})u^{(k)}(x)+\sum_{k=0}^{M}\frac{1}{k!}D^{M-1}(t^k\boldsymbol{b})\cdot\nabla u^{(k)}(x)\right.$$
$$\left.+\sum_{k=0}^{M}\frac{1}{k!}D^{M-1}(ct^k)u^{(k)}(x)\right\},$$

and

$$(3.3)\quad \nabla D^{M-1}F_M(t, x)=\nabla D^{M-1}f(t, x)+\nabla\Delta u^{(M-1)}(x)+t\nabla\Delta u^{(M)}(x)$$
$$-\Bigg[\sum_{k=2}^{M}\frac{1}{(k-1)!}\{u^{(k)}(x)\nabla D^{M-1}(at^{k-2})+\nabla u^{(k)}(x)D^{M-1}(at^{k-2})\}$$
$$+\sum_{k=0}^{M}\sum_{j=1}^{n}\frac{1}{k!}\{\nabla\partial_j u^{(k)}(x)D^{M-1}(t^k b_j)+\partial_j u^{(k)}(x)\nabla D^{M-1}(t^k b_j)\}$$
$$+\sum_{k=0}^{M}\frac{1}{k!}\{u^{(k)}(x)\nabla D^{M-1}(ct^k)+\nabla u^{(k)}(x)D^{M-1}(ct^k)\}\Bigg].$$

Let us prove our assertion for $M\geqq 5$. As was shown in Lemma 1.1, $u^{(k)}(x)$ with $k\leqq M$ is expressed as a linear form of $\{D^p f(0, x)$ with $p=0, 1, \cdots, M-2$, $\partial^\alpha D^q f(0, x)$ with $q=0, 1, \cdots, M-4$ and $|\alpha|\leqq 2$, and $\partial^\beta\phi(x)$ with $|\beta|\leqq(M-1)\}$. Therefore $\Delta u^{(k)}$ and $\nabla u^{(k)}(x)$ with $k\leqq M$ are linear forms of $\{\partial^\alpha D^p f(0, x)$ with $p=0, 1, \cdots, M-2$ and $|\alpha|\leqq 4$, and $\partial^\beta\phi(x)$ with $|\beta|\leqq(M+1)\}$. $\nabla\Delta u^{(k)}(x)$ and $\nabla\partial_j u^{(k)}(x)$ with $k\leqq M$ are linear forms of $\{\partial^\alpha D^p f(0, x)$ with $p=0, 1, \cdots, M-2$ and $|\alpha|\leqq 5$, and $\partial^\beta\phi(x)$ with $|\beta|\leqq(M+2)\}$. Thus it follows easily from (3.2) and (3.3) that for some positive constant C

$$|D^{M-1}F_M(t, x)|^2\leqq CG_M(f, \phi)(t, x)\quad\text{and}\quad|\nabla D^{M-1}F_M(t, x)|^2\leqq CH_M(f, \phi)(t, x)$$

hold. Insert the former inequality into

$$D^j F_M(t, x)=\int_0^t\frac{(t-s)^{M-2-j}}{(M-2-j)!}D^{M-1}F_M(s, x)ds$$

and the latter inequality into

$$\nabla F_M(t, x)=\int_0^t\frac{(t-s)^{M-2}}{(M-2)!}\nabla D^{M-1}F_M(s, x)ds,$$

respectively, where we used (1.5), and apply the Schwarz inequality, then we get the desired estimates.

We can argue similarly for $M\leqq 4$, too. Thus the proof is completed.

We set

$$(3.4)\qquad v(t, x)=u(t, x)-\sum_{k=0}^{M}\frac{t^k}{k!}u^{(k)}(x).$$

We define the energy $E_j(t, v)$ with $j=1, 2, 3$ of v by

$$E_1(t, v)=\frac{1}{2}\{\|Dv(t, \cdot)\|^2+\|\nabla v(t, \cdot)\|^2+\|v(t, \cdot)\|^2\},$$

$$E_2(t, v)=\frac{1}{2}\{\|D^2 v(t, \cdot)\|^2+\|\nabla Dv(t, \cdot)\|^2+\|Dv(t, \cdot)\|^2\}$$

and

$$E_3(t, v)=\frac{1}{2}\{\|\nabla Dv(t, \cdot)\|^2+\|\nabla^2 v(t, \cdot)\|^2+\|\nabla v(t, \cdot)\|^2\}.$$

We set

$$\lambda=\sup_{x\in \boldsymbol{R}^n} |a(0, x)|, \qquad \text{where} \quad a\in\mathscr{B}^\infty(\boldsymbol{R}\times\boldsymbol{R}^n).$$

LEMMA 3.2. (i) *Let* $u\in C^{M+1}(\Omega_T)\cap\mathscr{B}^{M+1}(I_T; H^1)$. *Then*

$$E_1(t, v)=O(t^{2M}) \quad \text{and} \quad E_2(t, v)=O(t^{2M-2}) \qquad \text{as} \quad t\to 0.$$

(ii) *Let* $u\in C^{M+1}(\Omega_T)\cap\mathscr{B}^{M+1}(I_T; H^2)$. *Then*

$$E_3(t, v)=O(t^{2M}) \qquad \text{as} \quad t\to 0.$$

(iii) *If* $M>\lambda$, *then* $\lim_{t\to 0}|t|^{-2\lambda}E_j(t, v)=0$ *for* $j=1$ *and* 3 *and if* $M>\lambda+1$, *then* $\lim_{t\to 0}|t|^{-2\lambda}E_2(t, v)=0$.

Proof. Noting that $D^k v(0, x)=0$ for $k=0, 1, \cdots, M$ and $D^{M+1}v(t, x)=D^{M+1}u(t, x)$, we get

$$v(t, x)=\int_0^t \frac{(t-s)^M}{M!} D^{M+1}u(s, x)ds, \qquad \nabla v(t, x)=\int_0^t \frac{(t-s)^M}{M!} \nabla D^{M+1}u(s, x)ds,$$

$$|\nabla^2 v(t, x)|=\left\{\sum_{j,k=1}^n \left|\int_0^t \frac{(t-s)^M}{M!}\partial_j\partial_k D^{M+1}u(s, x)ds\right|^2\right\}^{1/2},$$

$$Dv(t, x)=\int_0^t \frac{(t-s)^{M-1}}{(M-1)!} D^{M+1}u(s, x)ds, \quad \nabla Dv(t, x)=\int_0^t \frac{(t-s)^{M-1}}{(M-1)!} \nabla D^{M+1}u(s, x)\, ds$$

and

$$D^2 v(t, x)=\int_0^t \frac{(t-s)^{M-2}}{(M-2)!} D^{M+1}u(s, x)ds.$$

But, if $u\in\mathscr{B}^{M+1}(I_T; H^1)$, then the H^1-norm of the function $x\longmapsto D^{M+1}u(t, x)$ is bounded for $t\in I_T$. Hence by the Schwarz inequality we have

$$\begin{aligned}\int|\nabla v(t, x)|^2dx&=\sum_{k=1}^n\int dx\left|\int_0^t\frac{(t-s)^M}{M!}\partial_k D^{M+1}u(s, x)ds\right|^2\\&\leqq\frac{|t|}{(M!)^2}\sum_{k=1}^n\left|\int_0^t(t-s)^{2M}ds\int\left|\partial_k D^{M+1}u(s, x)\right|^2dx\right|\\&\leqq\frac{1}{(M!)^2(2M+1)}\|D^{M+1}u(s, \cdot)\|_1^2 t^{2M+2}.\end{aligned}$$

Similarly, for some positive constant C we have

$$\int|v(t, x)|^2dx\leqq Ct^{2M+2}, \quad \int|Dv(t, x)|^2dx\leqq Ct^{2M}, \quad \int|D^2v(t, x)|^2dx\leqq Ct^{2M-2},$$

$$\int |\nabla^2 v(t, x)|^2 dx \leqq Ct^{2M+2} \quad \text{and} \quad \int |\nabla Dv(t, x)|^2 dx \leqq Ct^{2M}.$$

Therefore we see that (i) is true if we insert the above estimates into the corresponding terms of E_1 and E_2. We can argue similarly for (ii), too.

(iii) follows immediately from (i) and (ii) because $2M-2>0$ for E_1 and E_3, and $2M-2-2>0$ for E_2.

LEMMA 3.3. *Let $\lambda=\sup_x|a(0, x)|$ and $\rho=(2\|Da\|_\infty+\|\boldsymbol{b}\|_\infty+\|c-1\|_\infty+1)$. Let M be an integer and*

$$M \geqq \begin{cases} \lambda+1 & \text{if } \inf_x \operatorname{Re} a(0, x)<0, \\ 2 & \text{if } \inf_x \operatorname{Re} a(0, x)\geqq 0. \end{cases}$$

Suppose that $\int G_M(f, \phi)(t, x)dx \leqq C$, where C is a positive constant independent of t. Suppose that u belongs to $C^{M+1}(\Omega_T)\cap\mathscr{B}^{M+1}(I_T; H^1)$. Then for any $t\in I_T$ the following estimates hold.

i) *If* $\inf_x \operatorname{Re} a(0, x)<0, \quad E_1(t, v)\leqq\dfrac{1}{2}|t|^{2\lambda}e^{\rho|t|}\left|\displaystyle\int_0^t |\tau|^{-2\lambda}e^{-\rho|\tau|}\|F_M(\tau, \cdot)\|^2 d\tau\right|.$

ii) *If* $\inf_x \operatorname{Re} a(0, x)\geqq 0, \quad E_1(t, v)\leqq\dfrac{1}{2}e^{\rho|t|}\left|\displaystyle\int_0^t e^{-\rho|\tau|}\|F_M(\tau, \cdot)\|^2 d\tau\right|.$

Proof. We have $(D^2v-\Delta v)D\bar{v}=LvD\bar{v}+F_M D\bar{v}$ and $(D^2\bar{v}-\Delta\bar{v})=\bar{L}\bar{v}Dv+\bar{F}_M Dv$, where $\bar{L}\bar{v}=-(\bar{a}/t)D\bar{v}-\bar{\boldsymbol{b}}\cdot\nabla\bar{v}-\bar{c}\bar{v}$. Since $u\in\mathscr{B}^{M+1}(I_T; H^1)$, v, Dv, D^2v and ∇v belong to $\mathscr{B}^0(I_T; L^2)$. By $Dv(0, x)=0$ we see that $(a/t)Dv$ also belongs to $\mathscr{B}^0(I_T; L^2)$. In conclusion $\Delta v=D^2v-Lv-F_M$ belongs to $\mathscr{B}^0(I_T; L^2)$. Therefore

$$\int \Delta v(t, x)D\bar{v}(t, x)dx \quad \text{and} \quad \int \Delta\bar{v}(t, x)Dv(t, x)dx$$

exist for any $t\in I_T$. Thus Green's integral formula gives

$$\begin{aligned} I_1 &= \int\{(D^2v(t, x)-\Delta v(t, x))D\bar{v}(t, x)+(D^2\bar{v}(t, x)-\Delta\bar{v}(t, x))Dv(t, x)\}dx \\ &= D\int|Dv|^2+D\int|\nabla v|^2=2\frac{dE_1}{dt}-2D\|v\|^2. \end{aligned}$$

On the other hand, noting that we can write $a(t, x)/t=(a(0, x)/t)+Da(t', x)$ for some t' between 0 and t, we get

$$\begin{aligned} I_2 &= \int(Lv(t, x)D\bar{v}(t, x)+\bar{L}\bar{v}(t, x)Dv(t, x))dx+\int(F_M(t, x)D\bar{v}(t, x)+\bar{F}_M(t, x)Dv(t, x))dx \\ &= -\frac{2}{t}\int(\operatorname{Re} a(0, x))|Dv(t, x)|^2dx-2\int(\operatorname{Re} Da(t', x))|Dv(t, x)|^2dx \end{aligned}$$

$$-2\mathrm{Re}\int \boldsymbol{b}(t, x)\cdot \nabla v(t, x) D\bar{v}(t, x)dx - 2\mathrm{Re}\int c(t, x)v(t, x)D\bar{v}(t, x)dx$$

$$+2\mathrm{Re}\int F_M(t, x)D\bar{v}(t, x)dx\,.$$

Since $I_1 = I_2$, we get

$$\begin{aligned}\frac{d}{dt}E_1(t, v) = &-(1/t)\int (\mathrm{Re}\, a(0, x))\,|Dv(t, x)|^2 dx \\ &-\mathrm{Re}\int Da(t', x)\,|Dv(t, x)|^2 dx - \mathrm{Re}\int \boldsymbol{b}(t, x)\cdot\nabla v(t, x)D\bar{v}(t, x)dx \\ &-\mathrm{Re}\int (c(t, x)-1)v(t, x)D\bar{v}(t, x)dx + \mathrm{Re}\int F_M(t, x)D\bar{v}(t, x)dx\,.\end{aligned}$$

Thus we have

$$\begin{aligned}(3.5)\quad \frac{d}{dt}E_1(t, v) \leqq &-(1/t)\int (\mathrm{Re}\, a(0, x))\,|Dv(t, x)|^2 dx + \|Da\|_\infty |Dv|^2 \\ &+\|\boldsymbol{b}\|_\infty \|\nabla v\|\,\|Dv\| + \|c-1\|_\infty \|v\|\,\|Dv\| + \|F_M\|\,\|Dv\| \\ \leqq &-(1/t)\int (\mathrm{Re}\, a(0, x))\,|Dv(t, x)|^2 dx + (1/2)\|F_M\|^2 + (\|c-1\|_\infty/2)\|v\|^2 \\ &+(1/2)(2\|Da\|_\infty + \|\boldsymbol{b}\|_\infty + \|c-1\|_\infty + 1)\|Dv\|^2 + (1/2)\|\boldsymbol{b}\|_\infty \|\nabla v\|^2 \\ \leqq &-(1/t)\int (\mathrm{Re}\, a(0, x)\,|Dv(t, x)|^2 dx + \rho E_1(t, v) + (1/2)\|F_M\|^2.\end{aligned}$$

Similarly we have

$$(3.6)\qquad \frac{d}{dt}E_1(t, v) \geqq -\frac{1}{t}\int (\mathrm{Re}\, a(0, x))\,|Dv(t, x)|^2 dx - \rho E_1(t, v) - \frac{1}{2}\|F_M\|^2\,.$$

We have from Lemma 3.1

$$(3.7)\qquad \|F_M(t, \cdot)\|^2 \leqq CC_M t^{2M-2}\,.$$

Now at first we shall prove i). Since $M \geqq \lambda + 1$, from (3.7) $|t|^{-2\lambda}e^{-\rho|t|}\|F_M(t, \cdot)\|^2$ is integrable in the interval between 0 and t. From (3.5) and (3.6) we get

$$\frac{d}{dt}E_1 \leqq \frac{2\lambda}{t}E_1 + \rho E_1 + \frac{1}{2}\|F_M\|^2 \quad \text{and} \quad \frac{d}{dt}E_1 \geqq \frac{2}{t}E_1 - \rho E_1 - \frac{1}{2}\|F_M\|^2\,.$$

Thus we get

$$\frac{d}{dt}(t^{-2\lambda}e^{-\rho t}E_1) \leqq \frac{1}{2}t^{-2\lambda}e^{-\rho t}\|F_M\|^2 \qquad \text{if} \quad t \geqq 0$$

and

$$\frac{d}{dt}((-t)^{-2\lambda}e^{\rho t}E_1) \geqq -\frac{1}{2}(-t)^{-2\lambda}e^{\rho t}\|F_M\|^2 \qquad \text{if} \quad t<0\,.$$

Integrating both sides of the above inequalities from 0 to t, we get

$$|t|^{-2\lambda}e^{-\rho|t|}E_1(t, v) \leqq \frac{1}{2}\left|\int_0^t |\tau|^{-2\lambda}e^{-\rho|\tau|}\|F_M(t, \cdot)\|^2 d\tau\right|,$$

where we used i) of Lemma 3.2. Thus we obtain

$$E_1(t, v) \leqq \frac{1}{2}|t|^{2\lambda}e^{\rho|t|}\left|\int_0^t |\tau|^{-2\lambda}e^{-\rho|\tau|}\|F_M(t, \cdot)\|^2 d\tau\right|,$$

which proves i).

Next we shall ii). Since $\operatorname{Re} a(0, x) \geqq 0$, we get from (3.5) and (3.6)

$$\frac{d}{dt}E_1 \leqq \rho E_1 + \frac{1}{2}\|F_M\|^2 \quad \text{if} \quad t \geqq 0 \quad \text{and} \quad \frac{d}{dt}E_1 \geqq -\rho E_1 - \frac{1}{2}\|F_M\|^2 \quad \text{if} \quad t<0\,.$$

Thus noting that $E_1(0, v)=0$, we get

$$E_1(t, v) \leqq \frac{1}{2}e^{\rho|t|}\left|\int_0^t e^{-\rho|\tau|}\|F_M(\tau, \cdot)\|^2 d\tau\right|.$$

Thus we have completed the proof of the lemma.

Now we prove a uniqueness theorem for the Cauchy problem (P-I).

THEOREM 3.4. *Let M be an integer such that*

$$M \geqq \begin{cases} \lambda+1 & \text{if} \ \inf_x \operatorname{Re} a(0, x)<0 \\ 2 & \text{if} \ \inf_x \operatorname{Re} a(0, x) \geqq 0\,. \end{cases}$$

Let u be a solution of the Cauchy problem (P-I). *Let v be a solution of the Cauchy problem* (P-II) *corresponding to* (P-I) *and be in $\mathscr{B}^1(I_T; H^1)$. Suppose that there exists a positive constant C such that, if* $\inf_x \operatorname{Re} a(0, x)<0$, *for* $t \in I_T$

$$E_1(t, v) \leqq \frac{1}{2}|t|^{2\lambda}e^{\rho|t|}\left|\int_0^t |\tau|^{-2\lambda}e^{-\rho|\tau|}\|F_M(\tau, \cdot)\|^2 d\tau\right| < C\,,$$

and if $\inf_x \operatorname{Re} a(0, x) \geqq 0$, *for* $t \in I_T$

$$E_1(t, v) \leqq \frac{1}{2}e^{\rho|t|}\left|\int_0^t e^{-\rho|\tau|}\|F_M(\tau, \cdot)\|^2 d\tau\right| < C\,.$$

Then u is the unique solution of the Cauchy problem (P-I) *in* $\mathscr{B}^1(I_T; H^1)$.

Proof. It is enough to show that if $u \in \mathscr{B}^1(I_T; H^1)$ is a solution of (P-I) with $f=\phi=0$, then $u=0$ in Ω_T. By $f=\phi=0$, we get from i) of Lemma 1.1 $u(0, x)=u^{(1)}(0, x)=\cdots=u^{(M)}(0, x)=0$. Hence we have from (3.1) $F_M=0$ in Ω_T. Therefore from

$\|F_M(\tau, \cdot)\|^2=0$ $E_1(t, u)=E_1(t, v)=0$. Thus we have shown that $u=0$ in Ω_T, which completes the proof.

In the following we shall show the estimates for $E_1(t, v)$, $E_2(t, v)$ and $E_3(t, v)$ which are necessary for the proof of the existence of a strong solution and its continuous dependence on the data.

LEMMA 3.5. *Suppose that M, $G_M(f, \phi)$ and u satisfy the same assumptions as in Lemma 3.3. Then*

$$E_1(t, v) \leqq A_M|t|^{2M-2}\left|\int_0^t\left|\int_0^\tau\int G_M(f, \phi)(s, x)dx\,ds\right|d\tau\right|^{1/2},$$

where

$$A_M=\begin{cases}\dfrac{\sqrt{C}\,C_M e^{\rho T}}{4\sqrt{M-1-\lambda}} & \text{if } \inf_x \operatorname{Re} a(0, x)<0\,,\\[2ex] \dfrac{\sqrt{C}\,C_M e^{\rho T}}{4\sqrt{M-1}} & \text{if } \inf_x \operatorname{Re} a(0, x)\geqq 0\,.\end{cases}$$

Proof. We shall discuss the case of $\inf \operatorname{Re} a(0, x)<0$. We suppose without loss of generality that $t \geqq 0$.

From Lemma 3.1 we have

$$\begin{aligned}
&\int_0^t \tau^{-2\lambda}e^{-\rho\tau}\|F_M(\tau, \cdot)\|^2 d\tau\\
&\quad\leqq \int_0^t \tau^{-2\lambda}\left\{C_M\tau^{2M-3}\int_0^\tau\int G_M(f, \phi)(s, x)dx\,ds\right\}d\tau\\
&\quad= C_M\int_0^t \tau^{2M-3-2\lambda}\left\{\int_0^\tau\int\cdots\right\}^{1/2}\left\{\int_0^\tau\int\cdots\right\}^{1/2}d\tau\\
&\quad\leqq \sqrt{C}\,C_M\int_0^t \tau^{2M-(5/2)-2\lambda}\left\{\int_0^\tau\int G_M(f, \phi)(s, x)dx\,ds\right\}^{1/2}d\tau\\
&\quad\leqq \sqrt{C}\,C_M\frac{t^{2M-2-2\lambda}}{2\sqrt{M-1-\lambda}}\left\{\int_0^t\int_0^\tau\int G_M(f, \phi)(s, x)dx\,ds\,d\tau\right\}^{1/2}.
\end{aligned}$$

Thus from i) of Lemma 3.3 we obtain

$$E_1(t, v) \leqq \frac{\sqrt{C}\,C_M e^{\rho T}}{4\sqrt{M-1-\lambda}}\,t^{2M-2}\left\{\int_0^t\int_0^\tau\int G_M(f, \phi)(\tau, x)dx\,ds\,d\tau\right\}^{1/2}.$$

We can discuss similarly the case $\inf \operatorname{Re} a(0, x) \geqq 0$, too. Thus we have completed the proof.

We set

$$J_M(f, \phi)(t)=\left|\int_0^t\left|\int_0^\tau\int G_M(f, \phi)(s, x)dx\,ds\right|d\tau\right|^{1/2}.$$

LEMMA 3.6. *Let M be an integer, and $M>\lambda+2$ if* $\inf \operatorname{Re} a(0, x)<0$ *and* $M\geqq 3$ *if* $\inf \operatorname{Re} a(0, x)\geqq 0$. *Suppose that u belongs to $C^{M+1}(\Omega_T)\cap \mathscr{B}^{M+1}(I_T; H^1)$ and there exists a positive constant C such that $\int G_M(f, \phi)(t, x)dx\leqq C$. Let $\sigma=\|a\|_\infty+3\|Da\|_\infty+\|\boldsymbol{b}\|_\infty+\|D\boldsymbol{b}\|_\infty+\|Dc\|_\infty+\|c-1\|_\infty+1$ and $\mu=2\{\|a\|_\infty+\|Da\|_\infty T^2+(\|D\boldsymbol{b}\|_\infty+\|Dc\|_\infty)T^4\}A_M$. Then, if* $\inf \operatorname{Re} a(0, x)<0$, *for any $t\in I_T$*

$$E_2(t, v)\leqq\frac{1}{2}|t|^{2\lambda}e^{\sigma|t|}\left|\int_0^t|\tau|^{-2\lambda}e^{-\sigma|\tau|}\{\mu J_M(f, \phi)(\tau)|\tau|^{2M-6}+\|DF_M(\tau, \cdot)\|^2\}\,d\tau\right|,$$

and if $\inf \operatorname{Re} a(0, x)\geqq 0$, *for any $t\in I_T$*

$$E_2(t, v)\leqq\frac{1}{2}e^{\sigma|t|}\left|\int_0^t e^{-\sigma|\tau|}\{\mu J_M(f, \phi)(\tau)|\tau|^{2M-6}+\|DF_M(\tau, \cdot)\|^2\}d\tau\right|.$$

Proof. Since $u\in\mathscr{B}^{M+1}(I_T; H^1)$, D^pv $(p=0, 1, 2, 3)$, $|\nabla Dv|$ and $|\nabla v|$ belong to $\mathscr{B}^0(I_T; L^2)$. Since $M\geqq 3$, $D^pv(0, x)=0$ $(p=0, 1, 2)$ holds. Therefore $(1/t)Dv(t, x)$ and $(1/t)D^2v(t, x)$ exist at $t=0$ and belong to $\mathscr{B}^0(I_T; L^2)$. Since $(1/t^4)\|Dv\|^2\leqq(2/t^4)E_1(t, v)$ and $M\geqq 3$, it follows from Lemma 3.5 that the function $(1/t^2)Dv(t, x)$ belongs to $\mathscr{B}^0(I_T; L^2)$. Hence

$$\Delta Dv(t, x)=\{D^3v+D(a/t)Dv+(a/t)D^2v+D(\boldsymbol{b}\cdot\nabla v)+D(cv)-DF_M\}(t, x)$$

belongs to $\mathscr{B}^0(I_T; L^2)$. Thus $\int|\Delta Dv(t, x)D^2v(t, x)|\,dx<\infty$.

We get easily

$$\begin{aligned}\{(D\Box v)D^2\bar{v}\}(t, x)=&-\frac{a(0, x)}{t}|D^2v(t, x)|^2-Da(t', x)|D^2v(t, x)|^2\\&+\frac{a(t, x)}{t^2}(DvD^2\bar{v})(t, x)-\frac{Da(t, x)}{t}(DvD^2\bar{v})(t, x)\\&-[\{(D\boldsymbol{b})\cdot\nabla v+\boldsymbol{b}\cdot\nabla Dv+vDc+cDv\}D^2\bar{v}](t, x)+(DF_MD^2\bar{v})(t, x),\end{aligned}$$

where t' is a number between 0 and t. Since the function $x\longmapsto Dv(t, x)$ belongs to H^2, we can perform the following integration by parts.

$$J_1=\int\{(D\Box v)D^2\bar{v}+(D\Box\bar{v})D^2v\}(t, x)dx=2\frac{d}{dt}E_2-D\|Dv\|^2.$$

On the other hand

$$\begin{aligned}J_2=&\int\{(DLv+DF_M)D^2\bar{v}+(D\bar{L}\bar{v}+D\bar{F}_M)D^2v\}(t, x)dx\\=&-\int\frac{2\operatorname{Re} a(0, x)}{t}|D^2v(t, x)|^2dx-2\int\operatorname{Re} Da(t', x)|D^2v(t, x)|^2dx\\&+2\operatorname{Re}\int\left(\frac{a}{t^2}DvD^2\bar{v}-\frac{Da}{t}DvD^2\bar{v}\right)dx-2\operatorname{Re}\int\{(D\boldsymbol{b}\cdot\nabla v)D^2\bar{v}+(\boldsymbol{b}\cdot\nabla Dv)D^2\bar{v}\}dx\end{aligned}$$

$$-2\mathrm{Re}\int\{(Dc)vD^2\bar{v}+(c-1)DvD^2\bar{v}\}dx+2\mathrm{Re}\int DF_M D^2\bar{v}dx-D\|Dv\|^2 .$$

Since $J_1=J_2$, by the Schwarz inequality we get

$$\begin{aligned}\frac{d}{dt}E_2\leqq &-\int\frac{\mathrm{Re}\,a(0,x)}{t}|D^2v(t,x)|^2dx+\|Da\|_\infty\|D^2v\|^2+\|D\boldsymbol{b}\|_\infty\|\nabla v\|\,\|D^2v\|\\ &+\|\boldsymbol{b}\|_\infty\|\nabla Dv\|\,\|D^2v\|+\|Dc\|_\infty\|v\|\,\|D^2v\|+\|c-1\|_\infty\|Dv\|\,\|D^2v\|\\ &+\|DF_M\|\,\|D^2v\|+J_3 ,\end{aligned}$$

where $J_3=(\|a\|_\infty/t^2)\|Dv\|\,\|D^2v\|+(\|Da\|_\infty/t)\|Dv\|\,\|D^2v\|$. Using Lemma 3.5,

$$\begin{aligned}2J_3&\leqq\|a\|_\infty\left(\frac{\|Dv\|^2}{t^4}+\|D^2v\|^2\right)+\|Da\|_\infty\left(\frac{\|Dv\|^2}{t^2}+\|D^2v\|^2\right)\\ &\leqq 2(\|a\|_\infty+\|Da\|_\infty t^2)t^{-4}E_1(t,v)+(\|a\|_\infty+\|Da\|_\infty)\|D^2v\|^2\\ &\leqq 2(\|a\|_\infty+\|Da\|_\infty T^2)A_M|t|^{2M-6}J_M(f,\phi)(t)+(\|a\|_\infty+\|Da\|_\infty)\|D^2v\|^2 .\end{aligned}$$

Here we remark that $2M-6\geqq 0$ because $M\geqq 3$. Thus we get

$$\begin{aligned}\frac{d}{dt}E_2(t,v)\leqq &-\int\frac{\mathrm{Re}\,a(0,x)}{t}|D^2v(t,x)|^2dx+\sigma E_2(t,v)+(\|D\boldsymbol{b}\|_\infty+\|Dc\|_\infty)E_1(t,v)\\ &+(\|a\|_\infty+\|Da\|_\infty T^2)A_M|t|^{2M-6}J_M(f,\phi)(t)+\frac{1}{2}\|DF_M\|^2 .\end{aligned}$$

Here applying Lemma 3.5 to $(\|D\boldsymbol{b}\|_\infty+\|Dc\|_\infty)E_1$, we get

$$\begin{aligned}(3.8)\quad \frac{d}{dt}E_2(t,v)\leqq &-\int\frac{\mathrm{Re}\,a(0,x)}{t}|Dv(t,x)|^2dx+\sigma E_2(t,v)\\ &+\{(\|D\boldsymbol{b}\|_\infty+\|Dc\|_\infty)T^4+\|a\|_\infty+\|Da\|_\infty T^2\}A_M|t|^{2M-6}J_M(f,\phi)(t)\\ &+\frac{1}{2}\|DF_M(t,\cdot)\|^2\\ =&-\int\frac{2\mathrm{Re}\,a(0,x)}{t}|Dv(t,x)|^2dx+\sigma E_2(t,v)\\ &+\frac{\mu}{2}J_M(f,\phi)(t)|t|^{2M-6}+\frac{1}{2}\|DF_M(t,\cdot)\|^2 .\end{aligned}$$

Similarly we get

$$\begin{aligned}(3.9)\qquad \frac{d}{dt}E_2(t,v)\geqq &-\int\frac{\mathrm{Re}\,a(0,x)}{t}|D^2v(t,x)|^2dx-\sigma E_2(t,v)\\ &-\frac{\mu}{2}J_M(f,\phi)(t)|t|^{2M-6}-\frac{1}{2}\|DF_M(t,\cdot)\|^2 .\end{aligned}$$

We have

$$J_M(f, \phi)(t) = \left| \int_0^t \left| \int_0^\tau \int G_M(f, \phi)(s, x) dx\, ds \right| d\tau \right|^{1/2} \leqq \sqrt{\frac{C}{2}}\, |t|$$

and therefore

$$|t|^{-2\lambda} J_M(f, \phi)(t) |t|^{2M-6} \leqq \sqrt{\frac{C}{2}}\, |t|^{2M-5-2\lambda}.$$

Consequently, noting that $2M-5-2\lambda > -1$, we have

$$\left| \int_0^t |\tau|^{-2\lambda} e^{-\sigma|\tau|} J_M(f, \phi)(\tau) |\tau|^{2M-6} d\tau \right| < \infty.$$

From Lemma 3.2 we have

$$\left| \int_0^t |\tau|^{-2\lambda} E_2(\tau, v) d\tau \right| < \infty \quad \text{and} \quad \lim_{t \to 0} |t|^{-2\lambda} E_2(t, v) = 0.$$

Thus by the same calculation as in Lemma 3.3 and from (3.8) and (3.9) we get

$$E_2(t, v) \leqq \frac{1}{2} |t|^{2\lambda} e^{\sigma|t|} \left| \int_0^t |\tau|^{-2\lambda} e^{-\sigma|\tau|} \{\mu J_M(f, \phi)(\tau) |\tau|^{2M-6} + \|DF_M(\tau, \cdot)\|^2\} d\tau \right|.$$

When $\inf_x \operatorname{Re} a(0, x) \geqq 0$ in particular, from (3.8) and (3.9) we have

$$\frac{d}{dt} E_2(t, v) \leqq \sigma E_2(t, v) + \left(\frac{\mu}{2}\right) J_M(f, \phi)(t) |t|^{2M-6} + \frac{1}{2} \|DF_M(t, \cdot)\|^2,$$

and

$$\frac{d}{dt} E_2(t, v) \geqq -\sigma E_2(t, v) - \frac{\mu}{2} J_M(f, \phi)(t) |t|^{2M-6} - \frac{1}{2} \|DF_M(t, \cdot)\|^2,$$

from which we easily get the corresponding estimate. Thus the proof is completed.

We set

$$K_M(f, \phi)(t) = C_M \left| \int_0^t \int H_M(f, \phi) dx\, ds \right|,$$

where C_M is the constant given in Lemma 3.1.

LEMMA 3.7. *Suppose that M satisfies the same assumption as in Lemma 3.3. Suppose that u belongs to $C^{M+1}(\Omega_T) \cap \mathscr{B}^{M+1}(I_T; H^2)$ and there exists a positive constant C such that*

$$\int G_M(f, \phi)(t, x) dx \leqq C \quad \textit{and} \quad K_M(f, \phi)(t) \leqq C.$$

Let

$$\kappa = 2\|Da\|_\infty + \|\nabla a\|_\infty + \|\boldsymbol{b}\|_\infty + \sum_{k=1}^n \|\partial_k b_k\|_\infty + \|c-1\|_\infty + \|\nabla c\|_\infty + 1$$

and $\nu = 2A_M(\|\nabla a\|_\infty + \|\nabla c\|_\infty T^2)$. *Then if* $M > \lambda + 1$

$$\text{i)}\quad E_3(t, v) \leqq \frac{1}{2}|t|^{2\lambda} e^{\kappa|t|}\left|\int_0^t |\tau|^{-2\lambda} e^{-\kappa|\tau|}\{\nu J_M(f, \phi)(\tau)|\tau|^{2M-4} + K_M(f, \phi)(\tau)|\tau|^{2M-3}\}d\tau\right|$$

and if in particular $\inf \operatorname{Re} a(0, x) \geqq 0$ *and* $M \geqq 2$,

$$\text{ii)}\quad E_3(t, v) \leqq \frac{1}{2} e^{\kappa|t|}\left|\int_0^t e^{-\kappa|\tau|}\{\nu J_M(f, \phi)(\tau)|\tau|^{2M-4} + K_M(f, \phi)(\tau)|\tau|^{2M-3}\}d\tau\right|.$$

Proof. Since $u \in \mathscr{B}^{M+1}(I_T; H^2)$, $D^2\partial_k v$, $\partial_k\partial_j v$, $\partial_k v \in \mathscr{B}^0(I_T; L^2)$ and $\partial_k Dv(0, x) = 0$. Hence the function $(1/t)D\partial_k v(t, x)$ belongs to $\mathscr{B}^0(I_T; L^2)$, and therefore

$$\int |\partial_k(\Delta v)\partial_k Dv|\,dx < \infty \qquad \text{for} \quad k = 1, 2, \cdots, n.$$

For $k = 1, 2, \cdots, n$ we have

$$\begin{aligned}\partial_k \Box v D\partial_k \bar{v} = &-\frac{a(0, x)}{t}|D\partial_k v(t, x)|^2 - Da(t', x)|D\partial_k v(t, x)|^2\\ &-\frac{\partial_k a}{t} Dv D\partial_k \bar{v} - (\partial_k \boldsymbol{b})\cdot \nabla v D\partial_k \bar{v} - \boldsymbol{b}\cdot\nabla(\partial_k v)D\partial_k \bar{v}\\ &-(\partial_k c)v D\partial_k \bar{v} - c\partial_k v D\partial_k \bar{v} + \partial_k F_M D\partial_k \bar{v},\end{aligned}$$

where t' is a number between 0 and t. Integrating by parts, we have

$$K_{k,1} = \int \{(\partial_k \Box v)D_k v + (\partial_k \Box \bar{v})D\partial_k v\,dx = D\{\|D\partial_k v\|^2 + \|\nabla\partial_k v\|^2 + \|\partial_k v\|^2\} - D\|\partial_k v\|^2.$$

On the other hand

$$\begin{aligned}K_{k,2} = &\int \{(\partial_k L v + \partial_k F_M)D\partial_k \bar{v} + (\partial_k \bar{L}\bar{v} + \partial_k \bar{F}_M)D\partial_k v\}dx\\ = &-\int \frac{2\operatorname{Re} a(0, x)}{t}|D\partial_k v(t, x)|^2 dx - 2\int \operatorname{Re} Da(t', x)|D\partial_k v(t, x)|^2 dx\\ &-2\operatorname{Re}\int \{(\partial_k \boldsymbol{b}\cdot\nabla v)D\partial_k \bar{v} + (\boldsymbol{b}\cdot\nabla\partial_k v)D\partial_k \bar{v}\}dx - 2\operatorname{Re}\int \{(\partial_k c)v D\partial_k \bar{v}\\ &+(c-1)\partial_k v D\partial_k \bar{v}\}dx + 2\operatorname{Re}\int \partial_k F_M D\partial_k \bar{v}\,dx - 2\operatorname{Re}\int \frac{\partial_k a}{t} Dv D\partial_k \bar{v}\,dx - D\|\partial_k v\|^2.\end{aligned}$$

Since

$$K_{k,1}=K_{k,2} \quad \text{and} \quad \sum_{k=1}^{n} K_{k,1}=2\frac{d}{dt}E_3-D\|\nabla v\|^2,$$

by the Schwarz inequality we have

$$\frac{d}{dt}E_3 \leqq -\int \frac{\operatorname{Re} a(0, x)}{t} |D\nabla v|^2 dx+\|Da\|_\infty \|D\nabla v\|^2$$

$$+\sum_{k=1}^{n} \|\partial_k b_k\|_\infty \|\nabla v\| \, \|D\nabla v\|+\|\boldsymbol{b}\|_\infty \|\nabla^2 v\| \, \|D\nabla v\|+\|\nabla c\|_\infty \|v\| \, \|D\nabla c\|$$

$$+\|c-1\|_\infty \|\nabla v\| \, \|D\nabla v\|+\frac{\|\nabla a\|_\infty}{t}\|Dv\| \, \|\nabla Dv\|+\|\nabla F_M\| \, \|D\nabla v\|$$

$$\leqq -\int \frac{\operatorname{Re} a(0, x)}{t} |D\nabla v(t, x)|^2 dx+\kappa E_3+(\|\nabla a\|_\infty+\|\nabla c\|_\infty t^2)\frac{E_1}{t^2}+\frac{1}{2}\|\nabla F_M\|^2.$$

From Lemma 3.5 we get $E_1(t, v)t^{-2} \leqq A_M |t|^{2M-4} J_M(f, \phi)(t)$ and from Lemma 3.1 we have $\|\nabla F_M(t, \cdot)\|^2 \leqq K_M(f, \phi)(t)|t|^{2M-3}$. Thus we get

$$\text{(3.10)} \qquad \frac{d}{dt}E_3(t, v) \leqq -\int \frac{\operatorname{Re} a(0, x)}{t} |D\nabla v(t, x)|^2 dx+\kappa E_3(t, v)$$

$$+\frac{\nu}{2}J_M(f, \phi)(t)|t|^{2M-4}+\frac{1}{2}K_M(f, \phi)(t)|t|^{2M-3}.$$

Similarly we get

$$\text{(3.11)} \qquad \frac{d}{dt}E_3(t, v) \geqq -\int \frac{\operatorname{Re} a(0, x)}{t} |D\nabla v(t, x)|^2 dx-\kappa E_3(t, v)$$

$$-\frac{\nu}{2}J_M(f, \phi)(t)|t|^{2M-4}-\frac{1}{2}K_M(f, \phi)(t)|t|^{2M-3}.$$

We have

$$\left| \int_0^t |\tau|^{-2\lambda} J_M(f, \phi)(\tau)|\tau|^{2M-4} d\tau \right| \leqq \sqrt{\frac{C}{2}} \left| \int_0^t |\tau|^{2M-3-2\lambda} d\tau < \infty \right.$$

because $2M-3-2\lambda>2(\lambda+1)-3-2\lambda=-1$ and $J_M(f, \phi)(t) \leqq \sqrt{C/2}\,|t|$, and we have

$$\left| \int_0^t |\tau|^{-2\lambda} K_M(f, \phi)(\tau)|\tau|^{2M-3} d\tau \right| = \left| \int_0^t K_M(f, \phi)(\tau)|\tau|^{2M-3-2\lambda} d\tau \right| < \infty.$$

From Lemma 3.2 we get

$$\lim_{t \to 0} |t|^{-2\lambda} E_3(t, v)=0 \quad \text{and} \quad \left| \int_0^t |\tau|^{-2\lambda} E_3(\tau, v) d\tau \right| < \infty.$$

Thus from (3.10) and (3.11) we obtain i). If in particular $\inf \operatorname{Re} a(0, x) \geqq 0$, we can omit

$$-\int\frac{\mathrm{Re}\,a(0,x)}{t}\,|D\nabla v(t,x)|^2dx$$

from (3.10) and (3.11), and therefore we obtain ii). Thus we have completed the proof.

§4. Existence of a solution for data in wider class

In §2 we proved that the Cauchy problem (P-I) had a unique solution in $\mathscr{B}^\infty(I_T;\mathscr{S})$ provided that $f\in\mathscr{B}^\infty(I_T;\mathscr{S})$ and $\phi\in\mathscr{S}$ where f and ϕ are the data of (P-I). In this section we assume that the pair (f,ϕ) is in some wider class than the product set $\mathscr{B}^\infty(I_T;\mathscr{S})\times\mathscr{S}$, which will be defined soon.

For $M=3, 4, 5, \cdots$ we define the space W_M of pairs of functions (f,ϕ) as follows. We let $W_M=\mathscr{B}^{M-1}(I_T;H^{M'})\times H^{M''+2}$, where

$$M'=\begin{cases}3 & \text{if } M=3\\ 5 & \text{if } M\geqq 4\end{cases},\quad\text{and}\quad M''=\begin{cases}5 & \text{if } M=4\\ M & \text{if } M=3 \text{ or } M\geqq 5\,.\end{cases}$$

We define the norm $|||f,\phi|||_M$ of an element $(f,\phi)\in W_M$ as follows. We let

$$|||f,\phi|||_M=\left\{\sum_{|\alpha|\leqq 1}\int_{-T}^{T}\|\partial^\alpha D^{M-1}f(t,\cdot)\|^2dt\right.$$
$$\left.+\sum_{|\alpha|\leqq M'}\sum_{p=0}^{M-2}\|\partial^\alpha D^p f(0,\cdot)\|^2+\sum_{|\alpha|\leqq M''+2}\|\partial^\alpha\phi\|\right\}^{1/2},$$

where M' and M'' are the same as in W_M. In the above definitions of norms all derivatives are taken in the strong sense. We set

$$V=\mathscr{B}^2(I_T;L^2)\cap\mathscr{B}^1(I_T;H^1)\cap\mathscr{B}^0(I_T;H^2)\,.$$

Our aim in this section is to show that (P-I) has a unique solution in V provided that (f,ϕ) is in a certain space W_M.

LEMMA 4.1. *Let (f,ϕ) be in W_M. Then there exists a sequence $\{f_j\}$ of functions in $\mathscr{B}^\infty(I_T;\mathscr{S})$ and a sequence $\{\phi_j\}$ of functions in $\mathscr{S}$ such that*

$$|||f-f_j,\phi-\phi_j|||_M\to 0\qquad\text{as } j\to\infty\,.$$

Proof. It is well-known that there exists a sequence $\{\phi_j\}$ of functions in $\mathscr{S}$ such that $\phi_j\to\phi$ as $j\to\infty$ in H^k.

Using the Friedrichs mollifier, we can make a sequence $\{g_j\}$ of functions in $C_0^\infty(\Omega_T)\subset\mathscr{B}^\infty(I_T;\mathscr{S})$ such that for $k=3$ or 5

$$\sum_{|\alpha|\leqq k}\sum_{p=0}^{M-1}\int_{-T}^{T}\|\partial^\alpha D^p(f-g_j)(t,\cdot)\|^2dt\to 0\qquad\text{as } j\to\infty\,. \tag{4.1}$$

It follows from (4.1) that there exists a subsequence $\{g_{i_j}\}$ of $\{g_j\}$ such that for almost all $t\in I_T$

$$\sum_{|\alpha|\leqq k}\sum_{p=0}^{M-2}\|\partial^\alpha D^p(f-g_{i_j})(t,\ \cdot)\|^2\to 0 \qquad \text{as} \quad j\to\infty\,. \tag{4.2}$$

Since $g_{i_j}\in C_0^\infty(\Omega_T)$ we can write

$$\partial^\alpha D^p\{g_{i_j}(0,\ x)-g(t,\ x)\}=\int_t^0 \partial^\alpha D^{p+1}g_{i_j}(\tau,\ x)d\tau\,,$$

and therefore we get

$$|\partial^\alpha D^p\{g_{i_j}(0,\ x)-g_{i_j}(t,\ x)\}|^2\leqq|t|\left|\int_t^0 |\partial^\alpha D^{p+1}g_{i_j}(\tau,\ x)|^2 d\tau\right|.$$

Thus we have

$$\sum_{|\alpha|\leqq k}\sum_{p=0}^{M-2}\|\partial^\alpha D^p\{g_{i_j}(0,\ \cdot)-g_{i_j}(t,\ \cdot)\}\|^2\leqq|t|\sum_{|\alpha|\leqq k}\sum_{p=0}^{M-2}\int_{-T}^{T}\|\partial^\alpha D^{p+1}g_{i_j}(\tau,\ \cdot)\|^2 d\tau\,.$$

From (4.1) we see that there is some positive constant C not depending on i_j such that

$$\sum_{|\alpha|\leqq k}\sum_{p=0}^{M-2}\int_{-T}^{T}\|\partial^\alpha D^{p+1}g_{i_j}(\tau,\ \cdot)\|^2 d\tau\leqq C\,.$$

Thus we have

$$\sum_{|\alpha|\leqq k}\sum_{p=0}^{M-2}\|\partial^\alpha D^p\{g_{i_j}(0,\ \cdot)-g_{i_j}(t,\ \cdot)\}\|^2\leqq C|t|\,. \tag{4.3}$$

Let ε be any small positive number. By (4.2), (4.3) and the fact that $f\in\mathscr{B}^{M-1}(I_T;\ H^k)$ we can take t such that

$$\left\{\sum_{|\alpha|\leqq k}\sum_{p=0}^{M-2}\|\partial^\alpha D^p\{g_{i_j}(0,\ \cdot)-g_{i_j}(t,\ \cdot)\|^2\right\}^{1/2}<\frac{\varepsilon}{3}\,,$$

$$\left\{\sum_{|\alpha|\leqq k}\sum_{p=0}^{M-2}\|\partial^\alpha D^p\{f(0,\ \cdot)-f(t,\ \cdot)\}\|^2\right\}^{1/2}<\frac{\varepsilon}{3}$$

and

$$\left\{\sum_{|\alpha|\leqq k}\sum_{p=0}^{M-2}\|\partial^\alpha D^p(f-g_{i_j})(t,\ \cdot)\|^2\to 0 \qquad \text{as} \quad j\to\infty\,.\right.$$

For this t we take a natural number N_0 such that for any $i_j>N_0$

$$\left\{\sum_{|\alpha|\leqq k}\sum_{p=0}^{M-2}\|\partial^\alpha D^p(f-g_{i_j})(t,\ \cdot)\|^2\right\}^{1/2}<\frac{\varepsilon}{3}\,.$$

Hence we get for any $i_j>N_0$

$$\left\{\sum_{|\alpha|\leqq k}\sum_{p=0}^{M-2}\|\partial^\alpha D^p(f-g_{i_j})(0,\ \cdot)\|^2\right\}^{1/2}<\varepsilon\,.$$

Therefore we obtain

$$\sum_{|\alpha|\leqq k}\sum_{p=0}^{M-2}\|\partial^\alpha D^p(f-g_{i_j})(0,\ \cdot)\|^2\to 0 \qquad \text{as} \quad j\to\infty\,.$$

Now we set $f_1=g_{i_1}, f_2=g_{i_2}, \cdots, f_n=g_{i_n}, \cdots$.

The above discussion proves that these $\{\phi_j\}$ and $\{f_j\}$ are desired sequences.

Let $u^{(k)}$ and $u_j^{(k)}$ $(k=0, 1, \cdots, M)$ be the functions given by means of Lemma 1.1, i) with respect to the given $(f, \phi)\in W_M$ and the sequence of pairs $\{(f_j, \phi_j)\}$ approximating (f, ϕ) in the above sense, respectively. Define

$$F_{M,j}(t, x)=f_j(t, x)-(\square-L)\sum_{k=0}^{M}\frac{t^k}{k!}u_j^{(k)}(x)$$

and

$$F_M(t, x)=f(t, x)-(\square-L)\sum_{k=0}^{M}\frac{t^k}{k!}u^{(k)}(x)\,.$$

Let v_j be the solution in $\mathscr{B}^\infty(I_T; \mathscr{S})$ of (P-II) with the datum $F_{M,j}$. In the following we shall show that there exists v in V such that $E_p(t, v_j-v)\to 0$ as $j\to\infty$ for $p=1, 2, 3$, which satisfies (P-II) with the datum F_M in the strong sense.

We set $\lambda=\sup|a(0, x)|$.

LEMMA 4.2. *Let*

$$M=\begin{cases}[\lambda]+3 & \text{if} \quad \inf \operatorname{Re} a(0, x)<0\\ \quad 3 & \text{if} \quad \inf \operatorname{Re} a(0, x)\geqq 0\,,\end{cases}$$

where $[\lambda]$ *denotes the integral part of* λ. *Suppose that* (f, ϕ) *is in* W_M. *Then there exists a solution* v *in* V *of* (P-II) *with the datum* F_M. *It satisfies the energy estimates with respect to* E_1, E_2 *and* E_3 *stated in Lemma* 3.3, *Lemma* 3.5, *Lemma* 3.6 *and Lemma* 3.7, *where all derivatives are taken in the strong sense.*

Proof. We shall write only the proof of the case of $\inf \operatorname{Re} a(0, x)<0$ and $M\geqq 5$. The proofs of other cases are similar to and rather simpler than that of the above case.

Let $\{f_j\}$ and $\{\phi_j\}$ be sequences in Lemma 4.1 such that $|||f-f_j, \phi-\phi_j|||_M\to 0$ as $j\to\infty$. Consider (P-II) with $F_{M,j}$ as a datum. Then there exists a solution v_j in $\mathscr{B}^\infty(I_T; \mathscr{S})$ by Lemma 2.10 and Corollary 2.12, which satisfies the energy estimates. We have

$$\begin{cases}(\square-L)(v_j-v_k)(t, x)=(F_{M,j}-F_{M,k})(t, x)\\ (v_j-v_k)(0, x)=D(v_j-v_k)(0, x)=0\,.\end{cases}$$

Therefore by Lemma 3.5, Lemma 3.6 and Lemma 3.7 we have

$$E_1(t, v_j-v_k)\leqq A_M|t|^{2M-2}\left|\int_0^t\left|\int_0^\tau\int G_M(f_j-f_k, \phi_j-\phi_k)(s, x)dx\,ds\right|d\tau\right|^{1/2},$$

$$E_2(t, v_j-v_k) \leqq \frac{1}{2}|t|^{2\lambda}e^{\sigma|t|}\left|\int_0^t |\tau|^{-2\lambda}e^{-\sigma|\tau|}\{\mu J_M(f_j-f_k, \phi_j-\phi_k)(\tau)|\tau|^{2M-6}\right.$$
$$\left. +\|D(F_{M,j}-F_{M,k})(\tau, \cdot)\|^2 d\tau\right|,$$

and

$$E_3(t, v_j-v_k) \leqq \frac{1}{2}|t|^{2\lambda}e^{\kappa|t|}\left|\int_0^t |\tau|^{-2\lambda}e^{-\kappa|\tau|}\{\nu J_M(f_j-f_k, \phi_j-\phi_k)(\tau)|\tau|^{2M-4}\right.$$
$$\left. +K_M(f_j-f_k, \phi_j-\phi_k)(\tau)|\tau|^{2M-3}\}d\tau\right|,$$

where G_M, J_M and K_M are defined in the previous section. Since $|||f_j-f_k, \phi_j-\phi_k|||_M \to 0$ as j and $k\to\infty$, by the definitions of G_M, $DF_{M,j}$, $DF_{M,k}$, J_M and K_M, we get $E_1(t, v_j-v_k)\to 0$ and $E_3(t, v_j-v_k)\to 0$ as j and $k\to\infty$. We take some positive constant C such that $|\int G_M(f_j-f_k, \phi_j-\phi_k)(\tau, x)dx|<C$. Therefore we get, noting $J_M \leqq \sqrt{C/2}|t|$ in the proof of Lemma 3.6,

$$\left|\int_0^t |\tau|^{-2\lambda}e^{-\sigma|\tau|}\mu J_M(f_j-f_k, \phi_j-\phi_k)(\tau, x)|\tau|^{2M-6}d\tau\right|$$
$$\leqq \frac{\sqrt{C}}{2}\mu\left|\int_0^t |\tau|^{2M-6-2\lambda+1}d\tau\right| = \frac{\sqrt{C}}{2}\mu\left|\int_0^t |\tau|^{1-2(\lambda-[\lambda])}d\tau\right| < \infty.$$

Hence we can apply the Lebesgue convergence theorem to the above integral and, $|||f_j-f_k, \phi_j-\phi_k|||_M\to 0$ and j and $k\to\infty$, we get $E_2(t, v_j-v_k)\to 0$ as j and $k\to\infty$. We can easily show that

$$\int_0^t |\tau|^{-2\lambda}e^{-\rho|\tau|}\|F_{M,j}(\tau, \cdot)-F_{M,k}(\tau, \cdot)\|^2 d\tau \to 0 \qquad \text{as } j \text{ and } k\to\infty,$$

which proves that v satisfies the estimate for E_1 in Lemma 3.3. Thus we have proved that there exists a function v such that i) $E_p(t, v-v_j)\to 0$ as $j\to\infty$ for $p=1, 2, 3$, ii) for almost all fixed t in I_T the function $x \longmapsto D^k v(t, x)$ is in H^{2-k} for $k=0, 1, 2$, iii) v satisfies the desired energy estimates. It is evident that v satisfies $(\Box - L)v = F_M$ and $v(0, x) = Dv(0, x) = 0$ in the strong sense.

We shall show the continuity of v and its derivatives in t. Let t_1 and t_2 be any numbers of I_T, and let $t_1<t_2$. We have

$$v_j(t_2, x)-v_j(t_1, x) = \int_{t_1}^{t_2} Dv_j(\tau, x)d\tau.$$

Thus

$$\|\Delta\{v_j(t_2, \cdot)-v_j(t_1, \cdot)\}\|^2 \leqq (t_2-t_1)\int_{t_1}^{t_2}\|\Delta Dv_j(\tau, \cdot)\|^2 d\tau \leqq 2(t_2-t_1)\int_{-T}^{T} E_3(\tau, v)d\tau.$$

Letting $j\to\infty$, we have

$$\|\Delta v(t_2, \cdot)-\Delta v(t_1, \cdot)\|^2 \leqq 2(t_2-t_1)\int_{-T}^{T} E_3(\tau, v)d\tau .$$

Thus we have shown that $t \longmapsto v(t, \cdot)$ is a continuous function in the L^2-norm. Similarly we can show that $t \longmapsto v(t, \cdot)$ is continuous in H^2, and $t \longmapsto Dv(t, \cdot)$ is continuous in H^1. Since $t \longmapsto (a(t, \cdot)/t)Dv(t, \cdot)$ is continuous in t except $t=0$ in the L^2-norm and the map $t \longmapsto D^2v(t, \cdot)=(\Delta+L)v(t, \cdot)+F_M(t, \cdot)$ is a continuous function of t except $t=0$ in the same norm. Since $E_2(t, v)\to 0$ as $t\to 0$, the map $t \longmapsto D^2v(t, \cdot)$ is continuous at $t=0$, too, in the L^2-norm. Thus $t \longmapsto D^2v(t, \cdot)$ is continuous in I_T in the L^2-norm. So it follows from the above argument that the map $t \longmapsto v(t, \cdot)$ is continuous in V.

Thus we have proved that v is a solution in V of (P-II).

When inf Re $a(0, x)\geqq 0$ or $M<5$, we similarly obtain the desired results.

Thus we have completed the proof of this lemma.

In the following we shall state our final result.

THEOREM 4.3. *Let*

$$M=\begin{cases}[\lambda]+3 & \text{if } \inf \operatorname{Re} a(0, x)<0 \\ 3 & \text{if } \inf \operatorname{Re} a(0, x)\geqq 0 .\end{cases}$$

Suppose that (f, ϕ) *is in* W_M. *Then* i) *there exists a unique solution* u *in* V *of* (P-I) *with data* f *and* ϕ. ii) *the mapping* $(f, \phi) \longmapsto u$ *is a continuous map from* W_M *into* V.

Proof. Let v be a solution of (P-II) stated in the preceding lemma. We define $u^{(0)}, u^{(1)}, \cdots, u^{(M)}$ for (f, ϕ) by (1.2) in § 1. We set

$$u(t, x)=v(t, x)+U_M(t, x), \qquad \text{where} \quad U_M(t, x)=\sum_{k=0}^{M} \frac{t^k}{k!} u^{(k)}(x).$$

From ii) of Lemma 1.1 U_M is in V. Hence this u is a solution of (P-I) in V.

Uniqueness is a direct consequence of Theorem 3.4.

By the definition of U_M it is easy to show that $(f, \phi) \longmapsto U_M$ is continuous from W_M into V. And it is clear from the energy estimates for v in Lemma 4.2 that $(f, \phi)\to v$ also is continuous from W_M into V. Thus we have shown the desired continuous dependence of u on (f, ϕ).

Thus we have completed the proof.

Acknowledgements. The author expresses his hearty thanks to Professor T. Yamanaka for giving useful advices and comments and critically reading the manuscript. The author is greatly indebted to Professor A. Matsumura for his suggestions concerning the transform of (P-I) into (P-II) by private communications.

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